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Conical Regularization for Abstract Constrained Optimization Problems in Hilbert Spaces

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Outline

Notations

- $(U, \langle \cdot, \cdot \rangle_U, \| \cdot \|_U)$ Hilbert Space
- $(Y, \langle \cdot, \cdot \rangle_Y, \| \cdot \|_Y)$ Hilbert Space
- $(H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)$ Hilbert Space
- $S : U \to H$ linear bounded operator
- $G : U \to Y$ linear bounded operator
- $C \subset Y$ closed, convex, pointed cone
- The positive dual cone of C :

$$C^+ = \{\lambda^* \in Y^* : \ \lambda^*(c) \geq 0, \quad \text{for every} \quad c \in C\}.$$

- The adjoint operator $G^* : Y \to U$ of the map G is defined by

$$\langle u, G^* y \rangle_U = \langle Gu, y \rangle_Y.$$

Model Problem

Consider the following convex optimization problem (P):

$$\begin{aligned} \text{Minimize} \quad & J(u) = \frac{1}{2}\|Su - z_d\|_H^2 + \frac{\kappa}{2}\|u - u_d\|_U^2, \\ \text{subject to} \quad & Gu \leq_C w, \quad u \in U. \end{aligned}$$

Here

- $z_d \in H$
- $u_d \in U$,
- $w \in Y$,
- $\kappa > 0$.

Under a feasibility assumption, the above problem is uniquely solvable and we denote by u_0 the unique solution.

Constraint Qualification and Regular Multipliers

- If $\mathrm{int}(C) \neq \emptyset$ and there exists $\hat{u} \in C$ such that

$$G(\hat{u}) - w \in -\mathrm{int}(C),$$

then u_0 can be computed by the optimality system:
There exist $u_0 \in U$ and $\mu_0 \in C^+$ such that

$$\begin{aligned} S^*(Su_0 - z_d) + \kappa(u - u_d) + G^*\mu_0 &= 0, \\ \langle \mu_0, G(u_0) - w \rangle_Y &= 0, \\ G(u_0) &\leq_C w. \end{aligned}$$

- The optimization problem (P) is said to be regular if u_0 can be obtained via a multiplier rule of above type.
- μ_0 is termed as a regular multiplier of (P).

Lack of Slater-type Constraint Qualification

In the literature, there are several variants of this constraint qualification which ensure the existence of multipliers.

- M.I. Henig, Proper efficiency with respect to cones. J. Optim. Theory Appl. 36 (1982) 387–407.
- A. Maugeri, F. Raciti, Remarks on infinite dimensional duality, J. Global Optimization, 46 (4), 2010.
- G. Isac, A.A. Khan, Dubovitskii-Milyutin approach in set-valued optimization, SIAM J. Control Optim. 47 (2008)
- J. Jahn, A. A. Khan, Generalized Contingent Epiderivatives in Set-valued Optimization: Optimality Conditions, Numerical Functional Analysis and Optimization, 28 (2002) 807–831. 144–162.

Meyer, Rösch, Tröltsch Regularization Approach (I)

Meyer, Rösch and Tröltsch (2006) proposed a novel regularization approach for the following optimization problem (Q):

$$\begin{aligned} \text{Minimize} \quad & \widetilde{J}(u) := \frac{1}{2}\|\widetilde{S}u\|_H^2 + \int_D \left(a(x)u(x) + \frac{\kappa}{2}u(x)^2 \right) dx, \\ \text{subject to} \quad & \widetilde{G}u \leq_C w, \\ & 0 \leq_C u, \quad u \in U, \end{aligned}$$

where $\widetilde{S} : U \to H$ and $\widetilde{G} : U \to Y$ are linear bounded operators.

- C. Meyer, A. Rösch, F. Tröltsch, *Optimal Control of PDEs with Regularized Pointwise State Constraints.* Comput. Optim. Appl. 33 (2006) 209–228.

Meyer, Rösch, Tröltsch Regularization Approach (II)

For $\varepsilon > 0$, define the regularized operator $\widetilde{G}_\varepsilon : U \to Y = U$ by

$$\widetilde{G}_\varepsilon := \widetilde{G} - \varepsilon I,$$

where I is the identity map.
Meyer, Rösch and Tröltsch considered the following family of regularized problems (Q_ε):

$$\begin{aligned} &\text{Minimize} && \widetilde{J}(u) := \frac{1}{2}\|\widetilde{S}u\|_H^2 + \int_D \left(a(x)u(x) + \frac{\kappa}{2}u(x)^2\right) dx, \\ &\text{subject to} && \widetilde{G}_\varepsilon(u) \leq_C w, \\ & && 0 \leq_C u, \quad u \in U. \end{aligned}$$

Meyer, Rösch, Tröltsch Regularization Approach (III)

Theorem: Assume that $U = Y = L^2(D)$ and $C = L^2_+(D)$. Assume that $\widetilde{S}$ and $\widetilde{G}$ are compact. Then:

- Problem (Q) has a unique solution u_0.
- For every $\varepsilon > 0$, there exists a unique solution u_ε of (Q_ε).
- Furthermore, if $\widetilde{G}$ is compact, then $u_{\varepsilon_n} \to u_0$ as $\epsilon_n \to 0^+$.
- If $-\widetilde{G}$ is non-negative, then (Q_ε) is regular for every $\varepsilon > 0$. That is, there exist regular multipliers μ^1_ε, $\mu^2_\varepsilon \in L^2_+(D)$.

Objective

Our primary objective is to devise a general regularization scheme for optimization problem (P) that possess the following characteristics.

- The regularized solutions should converge strongly to the solution of the original problem.
- The regularized problems should be regular, independent of the regularity of the original problem.
- The approach should be numerically feasible.

A General Family of Regularized Problems

Let $\delta > 0$. Given a set-valued map $F : [0, \delta) \rightrightarrows U$ with

$$F(0) = \{u \in U : Gu \leq_C w\},$$

we define a general one parameter regularization problem (P_ε):

$$\begin{aligned} &\text{Minimize} \quad && J(u) := \frac{1}{2}\|Su - z_d\|_H^2 + \frac{\kappa}{2}\|u - u_d\|_U^2, \\ &\text{subject to} \quad && u \in F(\varepsilon), \quad \varepsilon \in \mathbb{R}_+, \end{aligned}$$

where $F(\varepsilon)$ represents the feasible set of (P_ε) for $\varepsilon > 0$.

A General Result

Theorem: Assume that the following conditions hold:

- For every $\varepsilon \geq 0$, there exists unique solution u_ε of (P_ε).
- For every $\varepsilon > 0$, we have $F(0) \subset F(\varepsilon)$.
- The map F is weakly sequentially continuous at $\varepsilon = 0$. That is, for any $\{\varepsilon_n\} \subset \mathbb{R}_+$, $\{u_n\} \subset U$ such that $u_n \in F(\varepsilon_n)$, $\{\varepsilon_n\} \to 0^+$, $\{u_n\} \rightharpoonup \bar{u}$, we have $\bar{u} \in F(0)$.

Then for every $\{\varepsilon_n\} \to 0^+$, the sequence of solutions $\{u_{\varepsilon_n}\}$ of (P_{ε_n}) converges strongly to the solution of (P).

Conical Regularization

Our new approach is based on the notion of the dilating cones which we recall in the following:

Definition: Let $\delta > 0$. Given a closed, and convex cone $C \subset Y$, a family of solid, closed, and convex cones $\{C_\varepsilon\}_{0<\varepsilon\leq\delta}$ is said to be a family of dilating cones if it satisfies the following two conditions:

(i) $C\backslash\{0\} \subset int(C_\varepsilon)$, for every $0 < \varepsilon \leq \delta$.

(ii) $C = \bigcap_{0<\varepsilon} C_\varepsilon$.

- J.M. Borwein, D.M. Zhuang, *Super efficiency in vector optimization.* Trans. Amer. Math. Soc. 338 (1993) 105-122.

Conically Regularized Problems

We consider, the following family of conically regularized optimization problems (P_n):

$$\begin{aligned} \text{Minimize} \quad & J(u) := \frac{1}{2}\|Su - z_d\|_H^2 + \frac{\kappa}{2}\|u - u_d\|_U^2, \\ \text{subject to} \quad & Gu \leq_{C_n} w, \quad u \in U, \end{aligned}$$

where all the data is the same as in (P).

Solvability and Convergence

Theorem:

- For every $n \in \mathbb{N}$, the problem (P_n) is uniquely solvable.
- The sequence of regularized solutions $\{u_n\}$ converges strongly to the solution of (P).

Regularity of the Conically Regularized Problems

Theorem: Assume that the following feasibility holds:

$$\text{There exists } \tilde{u} \in U \text{ such that } G(\tilde{u}) \leq_C w \text{ and } G(\tilde{u}) \neq w.$$

Then (P_n) is regular for each $n \in \mathbb{N}$. That is, u_n is a minimizer of (P_n), if and only if, there exists Lagrange multiplier $\mu_n \in C_n^+$ such that

$$\begin{aligned} S^*(Su_n - z_d) + \kappa(u_n - u_d) + G^*\mu_n &= 0 \quad \text{in } U, \\ \langle \mu_n, Gu_n - w \rangle_Y &= 0 \\ Gu_n &\leq_{C_n} w. \end{aligned}$$

Furthermore,

$$G^*\mu_n \to -S^*(Su_0 - z_d) - \kappa(u_0 - u_d).$$

Regularity of the Original Problem

The following result shows that regularity of original problem can be achieved by the boundedness of multipliers associated to the regularized problems.

Theorem:

- Assume that the feasibility condition holds.
- Assume that the sequence $\{\mu_n\}$ is bounded.

Then (P) is regular.

The Finite Element Method: Basis Ideas

We aim to give a finite element discretization of a family of Henig dilating cones associated with the cone of positive functions of a general L_2-space.We follow three steps:

- In the first step, we give a characterization of the dual of a Henig dilating cone in a general normed space. We apply this characterization to the particular case of the cone of positive functions of a Lebesgue space.
- In the second step, we obtain a outer approximation of the Henig dilating cone of the unit simplex in $\mathbb{R}^n$ by means of a family of dilating cones associated with a basis of $\mathbb{R}^n$.
- Finally, in the last step, we give the finite-element discretization of the cone of positive function by relating a natural discretization of the Henig dilating cone with a determined cone associated to the unit simplex.

Henig Dilating Cone

Let $\Theta \subset C$ be a closed base of the cone C. That is, Θ is a closed, and convex subset such that

$$C = \{t\theta : t \geq 0, \ \theta \in \Theta\} \text{ and } 0 \notin cl\Theta.$$

For any

$$\epsilon \leq \epsilon_1 := \inf\{\|\theta\| : \theta \in \Theta\} > 0,$$

the associated Henig dilating cone is given by:

$$C_{\epsilon,\|\cdot\|}(\Theta) := cl(cone(\Theta + \epsilon B_{\|\cdot\|}))$$

It holds that

$$C = \bigcap_{\epsilon > 0} C_{\epsilon,\|\cdot\|}(\Theta)$$

Dual of the Dilating Cone

Theorem: The dual of the Henig dilating cone is given by:

$$C^{+}_{\epsilon,\|\cdot\|}(\Theta) = \{0\} \cup \{0 \neq \lambda^* \in C^{+} : \ \lambda^*(\theta) \geq \epsilon \|\lambda^*\| \text{ for every } \theta \in \Theta\}.$$

The Henig Dilating Cone of the Unit Simplex

The unit simplex in $Y = \mathbb{R}^m$ is defined by:

$$\Delta^m = \left\{(x_1, ..., x_m) \in \mathbb{R}^m_+ : \; x_1 + x_2 + \cdots + x_m = 1\right\}.$$

Δ^m is a closed and convex base for the cone $C = \mathbb{R}^m_+$.
The canonical bases of $\mathbb{R}^m$ will be denoted by $(e_i)_{i=1}^m \subset \mathbb{R}^m$ whereas associated family of biorthogonals by $(e_i^*)_{i=1}^m \subset (\mathbb{R}^m)^*$.
Define a family $\{\lambda^*_{\epsilon,1}, \lambda^*_{\epsilon,2}, \ldots, \lambda^*_{\epsilon,m}\} \subset (\mathbb{R}^m)^*$:

$$\lambda^*_{\epsilon,i}(\theta) = (1-\epsilon)e_i^* + \epsilon \sum_{j \neq i}^{m} e_j^*.$$

It is easy to show that this family is linearly independent provided that $\varepsilon < 1/2$.

Another Dilating Cone

We define

$$\widetilde{C}_\epsilon := \left\{ x \in \mathbb{R}^m : \lambda^*_{\epsilon,i}(x) \geq 0 \text{ for every } i = 1, \ldots, m \right\}.$$

It is evident from its definition that

$$C = \bigcap_{\epsilon > 0} \widetilde{C}_\epsilon.$$

Furthermore, we can give a computable matrix representation form,

$$\widetilde{C}_\epsilon = \{x \in \mathbb{R}^m : \ 0 \leq_{\mathbb{R}^m_+} L^m_\epsilon x\},$$

where $L^m_\epsilon \in \mathbb{R}^{m \times m}$ is the square matrix defined by

$$(L^m_\epsilon)_{ij} = \begin{cases} 1 - \varepsilon & \textit{if} \quad i = j \\ \varepsilon & \textit{if} \quad i \neq j. \end{cases}$$

An Important Inclusion

Theorem: Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^m$. Then for $\epsilon < \min\left\{\frac{1}{2}, \frac{2}{m}\right\}$, $\left\{\widetilde{C}_\epsilon\right\}_{\epsilon>0}$ is a family of solid, closed, and convex cones such that

$$C_{\epsilon,\|\cdot\|}(\Delta^m) \subset \widetilde{C}_\epsilon, \quad \text{for every } \epsilon > 0.$$

Finite Element Discretization

We take $Y = L^2(\Omega)$, $C = L^2_+(\Omega)$, and the base for C is

$$\Theta = \left\{ f \in L^2_+(\Omega) : \int_\Omega f(x)dx = 1 \right\}.$$

- Triangulation T^h of the domain Ω.
- $\{x_1, x_2, \ldots, x_m\}$ be the corresponding nodes.
- Y^h, the space of all linear continuous piecewise polynomials relative to T^h.
- The bases of Y^h will be denoted by

$$\{\varphi_1, \varphi_2, \ldots, \varphi_m\}.$$

- The space Y^h is then isomorphic to $\mathbb{R}^m$ and for any $u^h \in Y^h$, we define a vector $U \in \mathbb{R}^m$ by

$$U_i = u^h(x_i), \quad i = 1, 2, \ldots, m.$$

Discrete Dilating Cone

A natural discretization of the ordering cone $C = L^2_+(\Omega)$ is given by

$$C^h_+ = \{u^h : U_i \geq 0, \text{ for every } i = 1, \ldots, m\}.$$

Consequently, a discretized analogue of the Henig dilating cone $C_{\epsilon,\|\cdot\|_2}(\Theta)$ of $L^2_+(\Omega)$ in Y^h is given by the following cone:

$$C^h_\epsilon := cl\left(cone\left(\Theta^h + \epsilon B^h_{\|\cdot\|_2}\right)\right),$$

where the sets

$$\begin{aligned} \Theta^h &= \left\{v^h \in C^h_+ : \quad V^T \mathbb{Q}^h 1^m = 1\right\}, \\ B^h_{\|\cdot\|_2} &= \left\{b^h \in Y^h : \quad \sqrt{B^T \mathbb{Q}^h B} \leq 1\right\}, \end{aligned}$$

are the discretization of Θ and $B_{\|\cdot\|}$ of $L^2(\Omega)$.

Computable Formulas

Let $W^m \in \mathbb{R}^m$ be the vector defined by

$$W_i^m = \int_\Omega \varphi_i(s)ds = E_i^T Q^h 1^m, \quad \text{for every } i = 1, \ldots, m.$$

We set $\mathbb{D}^h = diag(W^m)$, and define a linear transformation $T : Y^h \to \mathbb{R}^m$ by

$$T(u^h) = \mathbb{D}^h U.$$

Theorem: Let d be the optimal value of the quadratic problem:

$$\begin{aligned} \text{Maximize} \quad & X^T (\mathbb{D}^h)^T \mathbb{D}^h X, \\ \text{subject to} \quad & X^T \mathbb{Q}^h X \leq 1, \quad X \in \mathbb{R}^m. \end{aligned}$$

Then for every suitable $\varepsilon \geq 0$, we have

$$T(C^h_{\varepsilon, \|\cdot\|_2}(\Theta^h)) \subset C_{\epsilon\sqrt{d}, \|\cdot\|}(\Delta^m).$$

Matrix Form

Theorem: If $u^h \in C^h_{\varepsilon,\|\cdot\|_2}(\Theta^h)$, then

$$0 \leq_{\mathbb{R}^m_+} L^m_{\varepsilon\sqrt{d}} \mathbb{D}^h U.$$

Numerical Experiments: Example 1

We choose $w(x) = \max\{\widehat{w}(x_1, x_2), -2\}$, $u_d = 0$, $z_d = 4 - \mu_0$ where

$$\widehat{w}(x_1, x_2) = 20(x_1 - 0.5)^2 + 20(x_2 - 0.5)^2 - 3.$$

The optimal solutions are $u_0 = y_0 = 2$ and the optimal multiplier is

$$\mu_0(x_1, x_2) = \max\{\widetilde{w}(x_1, x_2), 2\} - 2$$

ε	$\left\| u_\varepsilon^h - u_0 \right\|_2$	$\left\| y_\varepsilon^h - y_0 \right\|_2$
1.e-01	3.7138e-02	3.6999e-02
1.e-02	3.7136e-02	3.6998e-02
1.e-03	7.9946e-03	7.9686e-03
1.e-04	8.5119e-04	8.0619e-04
1.e-05	3.1016e-04	8.7168e-05

Numerical Experiments: Example 1

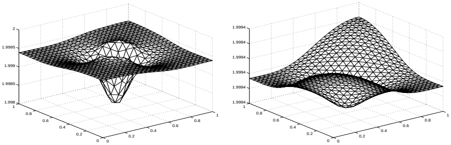

Figure: Control u_ϵ^h and state y_ϵ^h at $\epsilon = 10^{-5}$

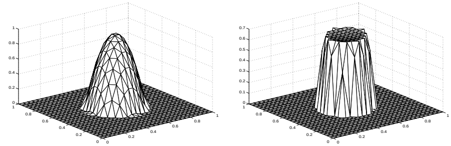

Figure: Multiplier μ_0. Multiplier μ_ϵ^h at $\epsilon = 10^{-5}$

Numerical Experiments: Example 2

We have $u_0 = y_0 \equiv 2$, $\kappa = 2$ and

$$w(x) = \begin{cases} -2x - 1 & \text{if} \quad x_1 < 0.5, \\ -2 & \text{if} \quad x_1 \geq 0.5, \end{cases}$$

$$z_d(x_1, x_2) = \begin{cases} x_1^2 - 0.5 & \text{if} \quad x_1 < 0.5, \\ 0.25 & \text{if} \quad x_1 = 0.5, \\ 0.75 & \text{if} \quad x_1 > 0.5. \end{cases}$$

$$u_d(x_1, x_2) = \begin{cases} 2.5 - x_1^2 & \text{if} \quad x_1 < 0.5, \\ 2.25 & \text{if} \quad x_1 \geq 0.5, \end{cases}$$

ε	$\left\Vert u_\varepsilon^h - u_0 \right\Vert_2$	$\left\Vert y_\varepsilon^h - y_0 \right\Vert_2$
1.e-01	1.0273e-01	8.5041e-02
1.e-02	5.1330e-02	2.2806e-02
1.e-03	4.1229e-02	2.1086e-03
1.e-04	4.0461e-02	2.2719e-03
1.e-05	3.9556e-02	2.6393e-03

Numerical Experiments: Example 2

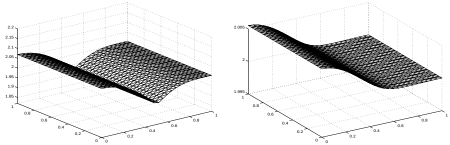

Figure: Control u_ϵ^h at $\epsilon = 10^{-5}$. State y_ϵ^h at $\epsilon = 10^{-5}$.

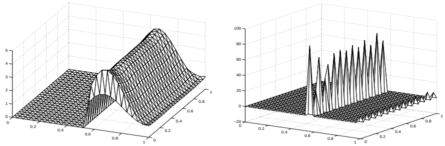

Figure: μ_ϵ^h at $\epsilon = 10^{-5}$ with m=121. μ_ϵ^h at $\epsilon = 10^{-5}$ m=676

Future Work

- Error Estimates for the Conical Regularization
- Comparison with Moreau-Yosida Regularization and Interior Point Methods.
- Optimality Conditions for Variational Inequalities by the Conical Regularization
- Extension to Optimization Problems with more General PDEs Constraints

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