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Some Remarks on Quasi Variational Inequalities

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Notations

- X-reflexive Banach space with X^* as its topological dual
- B-Hilbert space (the coefficient space)
- $A \subset B$, closed, convex (set of admissible coefficients)
- $C \subset X$, closed, convex
- $K : C \rightrightarrows C$ such that $K(x) \subset C$, closed, convex, for $x \in C$
- $F : X \rightrightarrows X^*$
- $T : B \times X \to X^*$
- $L : B \to X^*$
- $\rightharpoonup$ represents weak convergence

Quasi Variational Inequalities

- **Direct Problem:** Given $a \in A$, consider the QVI: find $x(a) = x \in C$ with $x \in K(x)$ such that for $w \in F(x)$, we have

$$\langle T(a,x) + w, z - x \rangle \geq \langle L(a), z - x \rangle, \quad \forall z \in K(x).$$

  [Bensoussan-Lions (1973,...), Mosco (1976,...), Kluge (1979), Baiocchi-Capelo (1984), Kravchuk-Neittaanmki (2007), Kenmochi (2009)].

- **Inverse Problem:** Given a measurement $\hat{z}$ of a solution x of the QVI, find a coefficient $a \in A$ such that $x(a)$ is closest to $\hat{z}$ in some norm.

Particular Cases

Case 1: Given a coefficient $a \in A$, find $x(a) = x \in C$ with $x \in K(x)$ such that for some $w \in F(x)$, we have

$$\langle w, z - x \rangle \geq \langle L(a), z - x \rangle, \quad \forall z \in K(x).$$

(Khan-Sama 2012.)

Case 2: Given a coefficient $a \in A$, find $x(a) = x \in C$ such that for some $w \in F(x)$, we have

$$\langle T(a, x) + w, z - x \rangle \geq \langle L(a), z - x \rangle, \quad \forall z \in C.$$

(Hoffmann-Sprekels 1984, Lenzen et al (2013)

Particular Cases Cont.

Case 3: Given a coefficient $a \in A$, find $x(a) = x \in C$ with $x \in K(x)$ such that

$$\langle T(a,x), z - x\rangle \geq \langle L(a), z - x\rangle, \quad \forall z \in K(x).$$

Case 4: Given a coefficient $a \in A$, find $x(a)$ such that

$$\langle T(a,x), z\rangle = \langle f, z\rangle, \quad \forall z \in X, \quad f \in X^*.$$

Solvability of QVIs

- For a fixed $v \in C$, consider parametric VI (PVI): find $x \in K(v)$ such that for some $w \in F(x)$, we have

$$\langle T(a,x) + w, z - x \rangle \geq \langle L(a), z - x \rangle, \quad \forall z \in K(v).$$

- Define the variational selection $S : C \rightrightarrows C$ such that

$$C \ni v \to S(v) = \{x | \quad x \text{ solves the PVI}\}.$$

- Notice that

$$x \in S(x) \Leftrightarrow x \text{ solves the QVI}$$

An Abstract Existence Result

THEOREM (Kluge 1979): Assume $C \subset F$ is convex and closed. Assume for $S : C \rightrightarrows C$, $S(u) \neq \emptyset$, closed and convex for every $u \in C$, and $graph(S)$ is weakly closed. Assume that $S(C)$ is bounded. Then S has at least one fixed point in C.

- $S(u) \neq \emptyset$ requires some sort of *coercivity* condition.

There exists $x_0 \in C$ such that $\forall x \in C$, $\forall\ w \in F(x)$

$$\frac{\langle T(a,x) + w, x - x_0 \rangle}{\|x\|} \to \infty \quad \text{as} \quad \|x\| \to \infty.$$

Maps of Monotone Type

- $F : X \rightrightarrows X^*$ is monotone if

$$\langle u - v, x - y \rangle \geq 0, \quad \forall\ u \in F(x),\ v \in F(y), \forall\ x, y$$

- $F : X \rightrightarrows X^*$ is maximal monotone if its graph is maximal
- $F : X \rightrightarrows X^*$ is strongly monotone if

$$\langle u - v, x - y \rangle \geq m\|x - y\|^2, \quad \forall\ u \in F(x),\ v \in F(y), \forall\ x, y$$

Maps of Monotone Type

Let $F : X \rightrightarrows X^*$ be a set-valued map.

- F is called pseudo-monotone, if for each $x \in X$, the set $F(x)$ is bounded, closed, and convex, F is finitely continuous, and for any $\{(x_n, w_n)\} \subset graph(F)$ such that $x_n \rightharpoonup x$ and

$$\limsup_{n\to\infty} \langle w_n, x_n - x \rangle \leq 0,$$

  it holds that for each $y \in X$, there exists $w(y) \in F(x)$ satisfying

$$\liminf_{n\to\infty} \langle w_n, x_n - y \rangle \geq \langle w(y), x - y \rangle.$$

- F is said to possess S_+ property if for any sequence $\{(x_n, w_n)\} \subset graph(F)$ with $x_n \rightharpoonup x \in dom(F)$ and $\limsup \langle w_n, x_n - x \rangle \leq 0$, we have $x_n \to x$.

Mosco Convergence

The map $K : C \rightrightarrows C$ is called continuous in Mosco sense, if the following conditions hold:

(M1) For any sequence $\{x_n\} \subset C$ with $x_n \rightharpoonup x$, and for each $y \in K(x)$, there exists $\{y_n\}$ such that $y_n \in K(x_n)$ and $y_n \to y$.

(M2) For $y_n \in K(x_n)$ with $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, we have $y \in K(x)$.

Minty Variational Inequality and Consequences

Theorem: Let $C \subset X$ be convex and closed, $F : X \rightrightarrows X^*$ be maximal monotone with $C \subset int(domain(F))$, ψ be proper, convex and lsc. Then the following are equivalent:

VI: There exists $x \in C$ such that for some $u \in F(x)$

$$\langle T(a,x) + u - f, z - x \rangle \geq \langle L(a), z - x \rangle \quad \forall\, z \in C.$$

MVI: For all $v \in F(z)$ and for all $z \in \Omega$

$$\langle T(a,z) + v - f, z - x \rangle \geq \langle L(a), z - x \rangle.$$

Consequence: The solution set of PVI is a closed and convex set. [Bakushinskii (1994), Alber et.al. (2003), Giannessi-Khan (2000)]

Weak Closedness of $Graph(S)$

$Graph(S)$ is weakly closed if:
For $x_n \rightharpoonup x$, $K(x_n)$ converges to $K(x)$ in Mosco sense, and additionally one of the following holds

- F is maximal monotone
- F is bounded and pseudo-monotone
- F is bounded and generalized monotone

Existence by the Contraction Principle

This approach imposes restriction on the set-valued map K. For instance

- K has the following structure

$$K(x) = m(x) + C.$$

  (Fortunately, in many applications this structure exists.)

- K satisfy the following condition

$$\|P_{K(x)}(z) - P_{K(y)}(z)\| \leq \ell \|x - y\|$$

- Requires strong monotonicity and Lipschitz continuity of the involved maps.

Existence by the Contraction Principle Cont.

For some applications, it is convenient to consider the following QVI: find $x(a) = x \in X$ such that for some $w \in F(x)$ and for every $z \in X$, we have

$$\langle T(a,x) + w, z - x \rangle \geq \langle L(a), z - x \rangle + \varphi(x,x) - \varphi(x,z).$$

In this case, suitable assumptions can be imposed on the map φ.

Inverse Problem Formulation

We introduce the following cost function

$$J(a, x(a)) := \|x(a) - z\|^2 + \epsilon\|a\|^2,$$

where $\epsilon > 0$ and $x(a)$ is a solution of the QVI: find $x(a) = x \in C$ with $x \in K(x)$ such that for some $w \in F(x)$, we have

$$\langle T(a, x) + w, y - x\rangle \geq \langle L(a), y - x\rangle, \quad \forall y \in K(x).$$

Here z is a measurement of a solution of the QVI.
We pose the inverse problem

$$\min_{a \in A} J(a, x(a)).$$

Remark: Sometimes it is convenient to consider the function:

$$J(a, x(a)) := \|\Gamma(x(a)) - z\|_W^2 + \epsilon\|a\|^2.$$

Solvability of the Inverse Problem

THEOREM: Assume that:

- For every $a \in A$, the state quasi variational inequality has a solution.
- $T(a, \cdot)$ is monotone (uniformly in a).
- For $a_n \rightharpoonup a$, $x_n \to x$, $T(a_n, x_n) \to T(a, x)$.
- Assume that F is bounded, pseudo-monotone with S_+ property, and satisfies the coercivity condition (uniformly in a): there exists $x_0 \in C$ such that $\forall x \in C$, $\forall\ w \in F(x)$

$$\frac{\langle T(a,x) + w, x - x_0 \rangle}{\|x\|} \to \infty \quad \text{as} \quad \|x\| \to \infty.$$

- Assume that the map $v \to K(v)$ is M-continuous.

Then the inverse problem has a solution $(a, x(a))$.

Semi-Monotone vs Pseudo-Monotone QVI

A map $\tilde{F} : X \times X \to X^*$ is called semi-monotone if,

(SM1) For any fixed $z \in X$, the mapping $x \to \tilde{F}(z, x)$ is maximal monotone.

(SM2) For any $\{z_n\} \subset X$ with $z_n \rightharpoonup z \in X$ and for every $z^* \in \tilde{F}(z, x)$, there exists a sequence $\{z_n^*\}$ such that $z_n^* \in \tilde{F}(z_n, x)$ and $z_n^* \to z^*$.

Furthermore, given a semi-monotone map $\tilde{F} : X \times X \to X^*$, the map $F : X \to X^*$ defined by $F(x) = \tilde{F}(x, x)$, for all $x \in X$ is called a map generated by $\tilde{F}$.

Convergence for a Contaminated Data

We next study the convergence behavior of the inverse problem for contaminated data.

For a bounded map $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$, we assume

- For any $x \in C$, and for every $w \in F(x)$ (respectively $w_n \in F(x)$), there exists $\tilde{w} \in F_n(x)$ (respectively $w \in F(x)$) satisfying

$$\|w - \tilde{w}\| \leq \alpha_n \kappa(\|x\|), \quad \alpha_n > 0, \quad \alpha_n \to 0.$$

Inverse Problem for Perturbed QVI

We introduce the following *perturbed* cost function:

$$J_n(a, x_n(a)) := \|\Gamma x_n(a) - z\|^2 + \epsilon\|a\|^2,$$

where $x_n = x_n(a)$ is a solution that corresponds to the coefficient a through the following perturbed quasi variational inequality: find $x_n \in K(x_n)$ such that for some $w_n \in F_n(x_n)$, we have

$$\langle w_n, z - x_n\rangle \geq \langle L(a), z - x_n\rangle, \quad \text{for every } z \in K(x_n).$$

Solvability of the Inverse Problem

THEOREM: Assume that the following conditions hold:

- Assume that for every $a \in A$, perturbed QVIs are solvable.
- Assume that $F : X \rightrightarrows X^*$ is maximal monotone.
- $C \subset int(domain(F))$,
- $K : C \rightrightarrows C$ is M-continuous.
- Assume that for every $a \in A$, the QVI is uniquely solvable.

Then, for every $n \in \mathbb{N}$ the perturbed inverse problem has a solution (a_n, x_n), and there exists a subsequence $\{(a_n, x_n)\}$ that converges weakly to a solution.

Quasi Variational Inequalities Revisited

X-reflexive Banach space, $C \subset H$ closed and convex, $f \in X^*$

$K : C \to 2^C$ a set-valued map, $K(x)$ closed and convex $\forall\ x \in C$

$F : X \rightrightarrows X^*$ a set-valued map

$\psi : X \to R$ proper, convex and lower-semicontinuous (lsc)

QVI: Find $x \in C$ such that $x \in K(x)$ and for some $w \in F(x)$

$$\langle w - f, z - x \rangle \geq \psi(x) - \psi(z) \quad \text{for all } z \in K(x).$$

Objective: Regularization of Multi-Valued QVI?

Regularization for QVI serves (at least) two purposes:

1. Noncoercive QVI

Examples:

- Sandpile model as non-coercive parabolic QVI. See Prigozhin (2004).
- Non-coercive obstacle problems. See Garroni-Gossez (1983).

2. Noisy Data

Examples:

- QVI as optimality condition for some inverse problems

Regularized QVI

Regularized QVI: Find $x_n \in C$ such that $x_n \in K(x_n)$ and for some $u_n \in F_n(x)$ and for all $z \in K(x_n)$

$$\langle u_n + \epsilon_n x_n - f_n, z - x_n \rangle \geq \psi_n(x) - \psi_n(z).$$

[Bruckner (1981), Giannessi-Khan (2000), Khan-Raciti-Rouhani (2010)].

Regularization of VI: Lions-Stampacchia (1967), Browder (1966,...), Browder-Ton (1969,...), Mosco (1969,...), Bakushinskii (1979,...), Liskovets (1981,...), Gwinner (1997), Konnov (2007,...), Kaplan-Tichatschke (1994,...), Liu-Nashed (1998), Nashed-Scherzer (1999,...), Isac (1997), Ryazantseva (1981,...), Alber-Butnariu-Ryazantseva (2001,...), Khan-Raciti-Rouhani (2008)...

Regularized Solutions

Assume that the perturbed data is similar to the exact data, except may be the coercivity.

Verify that F_n is coercive (with respect to ψ_n)

- There exists $z_n \in C$ such that $\psi_n(z_n) < c < \infty$ and for every $x \in K$ and for every $u_n \in F_n(x)$

$$\langle u_n, x - z_n \rangle + \psi_n(x) - \psi_n(z_n) \geq -\|x\| a_n(\|x\|),$$

where

- $\limsup_{t \to \infty} a_n(t) \leq c < \infty$
- $\limsup_{t \to \infty} a_n(t) = \infty, \quad \dfrac{t}{a_n(t)} \to \infty$ as $t \to \infty$.

Bounded Regularized Solutions: Monotone Case

THEOREM: Assume that

- $\|f_n - f\| \leq c\beta_n \quad (0 < \beta_n \downarrow 0)$
- $|\psi_n(x) - \psi(x)| \leq c\gamma_n\kappa(\|x\|) \quad (0 < \gamma_n \downarrow 0)$
- For any $w \in F(x)$ there exists $w_n \in F_n(x)$ such that

$$\|w_n - w\| \leq c\alpha_n\kappa(\|x\|) \quad (0 < \alpha_n \downarrow 0).$$

- For any $w_n \in F_n(x))$ there exists $w \in F(x)$ such that

$$\|w_n - w\| \leq c\alpha_n\kappa(\|x\|) \quad (0 < \alpha_n \downarrow 0).$$

Assume that the sequence $\{x_n\}$ of regularized solutions is bounded. Then there exists a subsequence converging weakly to a solution of QVI.

Bounded Regularized Solutions: Pseudo Monotone Case

THEOREM: Assume that

- F_n is bounded.
- $\|f_n - f\| \leq c\beta_n \quad (0 < \beta_n \downarrow 0)$
- $|\psi_n(x) - \psi(x)| \leq c\gamma_n\kappa(\|x\|) \quad (0 < \gamma_n \downarrow 0)$
- For $w \in F(x)$ and $w_n \in F_n(x)$ we have

$$\|w_n - w\| \leq c\alpha_n\kappa(\|x\|) \quad (0 < \alpha_n \downarrow 0).$$

Assume that the sequence $\{x_n\}$ of regularized solutions is bounded. Then there exists a subsequence converging weakly to a solution of QVI.

Strong Convergence: Monotone Case

Ingredient: Fast Mosco Convergence of $K(x_n)$ to $K(x)$ w.r.t e_n

THEOREM: Assume that

- $\|f_n - f\| \leq c\beta_n \quad (\frac{\beta_n}{e_n} \downarrow 0)$
- $|\psi_n(x) - \psi(x)| \leq c\gamma_n\kappa(\|x\|) \quad (\frac{\gamma_n}{e_n} \downarrow 0)$
- For any $w \in F(x)$ there exists $w_n \in F_n(x)$ such that

$$\|w_n - w\| \leq c\alpha_n\kappa(\|x\|) \quad (\frac{\alpha_n}{e_n} \downarrow 0).$$

- For any $w_n \in F_n(x))$ there exists $w \in F(x)$ such that

$$\|w_n - w\| \leq c\alpha_n\kappa(\|x\|) \quad (\frac{\alpha_n}{e_n} \downarrow 0).$$

Assume that the sequence $\{x_n\}$ of regularized solutions is bounded. Then there exists a subsequence converging strongly to a solution

Iterative Regularization for QVI

For variational inequalities Bakushiinskii-Polak (1979) used the triangle inequality:

$$\|x_{n+1} - x^*\| \leq \|x_{n+1} - x_{e_n}\| + \|x_{e_n} - x^*\|$$

where x_{n+1} is an iterative solution by some suitable scheme. If $K(x) = C + M(x)$ then QVI can be solved by:

$$x_{n+1} = M(x_n) + PC(x_n - \alpha_n(F_n(x_n) + e_n x_n) - M(x_n))$$

Unfortunately, the standard techniques of using the recursive inequalities won't work here!

QVI as a Minimization Problem: Generalized Solutions

For $v \in C$, consider the following parametric VI: find $x \in K(v)$ such that for some $w \in F(x)$, we have

$$\langle w, z - x \rangle \geq 0, \quad \text{for every } z \in K(v).$$

Define $S : C \rightrightarrows C$ such that for any $v \in C$, $S(v)$ is the set of all solutions of PVI.
We now pose the following minimization problem (PIP): find $(x, u) \in Graph(S)$ such that

$$\|x - u\|^2 \leq \|y - v\|^2 \quad \text{for every } (y, v) \in Graph(S).$$

If $(u, x) \in Graph(S)$ is a minimizer of PIP, then $x \in C$ is called a generalized solution of the QVI.

Generalized Solutions Cont.

The following connections between a generalized solution and a classical solution are immediate:

- If PIP is solvable with (x, u) as a solution, and $\|x - u\| = 0$, then QVI is solvable.
- If QVI is solvable, then PIP is also solvable, and their solution sets coincide.

We recall
LEMMA (Bruckner 1981): Assume that there exists $(\tilde{x}, \tilde{u}) \in Graph(S)$ such that the set

$$\Phi = \{(y, v) \in Graph(S) | \quad \|y - v\| \leq \|\tilde{x} - \tilde{u}\|\}$$

is weakly compact. Then the QVI has a generalized solution.

Implicit Obstacle problem

Let L be an elliptic operator given by

$$L \equiv -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left[a_{i,j}\frac{\partial}{\partial x_i}\right] + \sum_{i=1}^{2} b_i \frac{\partial}{\partial x_i} + d,$$

where the coefficients are in L^∞. Let $M : L^\infty \to L^\infty$ be a map given by

$$M(\phi)(x) = 1 + \inf_{x+\zeta\in\Omega,\zeta\geq 0} \phi(x+\zeta),$$

and let $f \in L^\infty$ be such that $f \geq 0$ a.e. in Ω. With this data we consider the following obstacle problem: find $u \in H^1(\Omega)$ such that

$$\begin{aligned} Lu &\leq f, \quad \text{in } \Omega \quad u \leq M(u), \quad \text{in } \Omega \\ (Lu-f)(u-Mu) &= 0, \quad \text{in } \Omega \quad \frac{\partial u}{\partial v_\alpha} \leq 0, \quad \text{on } \Omega \\ (u-Mu)\frac{\partial u}{\partial v_\alpha} &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Here ∂v_α is the conormal derivative.

Future Work

- Penalty Method
- Optimality Conditions
- Numerical Experiments
- Detailed Applications

References

1. S. Adly, M. Bergounioux and M.A. Mansour, Optimal control of a quasi variational obstacle problem, *J. Global Optim.*, **47** (2010), 421–435.
2. C. Baiocchi and A. Capelo, *Variational and quasivariational inequalities. Applications to free boundary problems,* John Wiley & Sons, Inc., New York, 1984.
3. A. Bensoussan and J.L. Lions, Nouvelles mthodes en contrle impulsionnel, *Appl. Math. Optim.*, **1**, (1974/75), 289–312.
4. J.W. Barrett and L. Prigozhin, A quasi-variational inequality problem in superconductivity, Math. Models Methods Appl. Sci., **20** (2010), 679-706.
5. H. Dietrich, Optimal control problems for certain quasivariational inequalities, Optimization, **49** (2001), 67–93.
6. M Hintermüller, Inverse coefficient problems for variational inequalities: optimality conditions and numerical realization, *M2AN Math. Model. Numer. Anal.,* **35** (2001), 129–152.
7. R. Kano, N. Kenmochi and Y. Murase, Existence theorems for elliptic quasi variational inequalities in Banach spaces. Recent advances in nonlinear analysis, 149–169, World Sci. Publ., Hackensack, NJ, 2008.
8. Z. Liu, Generalized quasi variational hemi-variational inequalities, *Appl. Math. Lett.*, **17** (2004), 741–745.
9. U. Mosco, Implicit variational problems and quasi variational inequalities. In: *Nonlinear operators and the calculus of variations,* pp. 83–156. *Lecture Notes in Math.*, Vol. 543, Springer, Berlin, 1976.
10. F. Patrone, On the optimal control for variational inequalities, *J. Optim. Theory Appl.,* **22** (1977), 373-388.