european social fund in the czech republic

EUROPEAN UNION

MINISTRY OF EDUCATION, YOUTH AND SPORTS

OP Education for Competitiveness

INVESTMENTS IN EDUCATION DEVELOPMENT

Streamlining the Mathematics Studies at the Faculty of Science of Palacky University in Olomouc

Registry number: CZ.1.07/2.2.00/28.0141

Numerical Treatment of Singular BVPs in ODEs

Ewa Weinmüller

Analysis and Scientific Computing, Vienna University of Technology

Palacky University, Olomouc, Czech Republic, 23 – 25 June 2014

TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Contents

Examples: Convergence of the forward Euler

Example 1: Consider the linear problem

$$y''(t) = -\frac{2}{t}y'(t) \underbrace{-n^2\cos(nt) - \frac{2}{t}n\sin(nt)}_{\text{has a limit for } t\to 0}, \quad t \in (0,1],$$

$$y(0) = 2, \quad y'(0) = 0,$$

where $y(t) = 1 + \cos(nt), \; n = 3.$

Examples: Convergence of the forward Euler

Example 1: Consider the linear problem

$$y''(t) = -\frac{2}{t} y'(t) \underbrace{- n^2 \cos(nt) - \frac{2}{t} n \sin(nt)}_{\text{has a limit for } t \to 0}, \quad t \in (0,1],$$

$$y(0) = 2, \quad y'(0) = 0,$$

where $y(t) = 1 + \cos(nt), \ \ n = 3.$

Example 2: Consider the nonlinear problem

$$y''(t) = -\frac{2}{t} y'(t) - y^5(t), \quad t \in (0,1],$$
$$y(0) = 1, \quad y'(0) = 0,$$

where $y(t) = \frac{1}{\sqrt{1+t^2/3}}$.

Example 1

h	$\|\varepsilon_h\|$	p	c
$1/5$	8.5	0.970	$-4.0 \cdot 10^{+01}$
$1/5 \cdot 2^{-1}$	4.3	0.982	$-4.1 \cdot 10^{+01}$
$1/5 \cdot 2^{-2}$	$2.1 \cdot 10^{-01}$	0.990	$-4.2 \cdot 10^{+01}$
$1/5 \cdot 2^{-3}$	$1.1 \cdot 10^{-01}$	0.995	$-4.3 \cdot 10^{+01}$
$1/5 \cdot 2^{-4}$	$5.5 \cdot 10^{-01}$	0.997	$-4.3 \cdot 10^{+01}$
$1/5 \cdot 2^{-5}$	$2.7 \cdot 10^{-02}$	0.998	$-4.4 \cdot 10^{+01}$
$1/5 \cdot 2^{-6}$	$1.3 \cdot 10^{-02}$	0.999	$-4.4 \cdot 10^{+01}$
$1/5 \cdot 2^{-7}$	$6.9 \cdot 10^{-02}$	0.999	$-4.4 \cdot 10^{+01}$
$1/5 \cdot 2^{-8}$	$3.4 \cdot 10^{-02}$	0.999	$-4.4 \cdot 10^{+01}$
$1/5 \cdot 2^{-9}$	$1.7 \cdot 10^{-03}$	0.999	$-4.4 \cdot 10^{+01}$

Example 2

h	$\|\|\varepsilon_h\|\|$	p	c
$1/5$	$2.4 \cdot 10^{-02}$	0.834	$-9.2 \cdot 10^{-01}$
$1/5 \cdot 2^{-1}$	$1.3 \cdot 10^{-02}$	0.921	$-1.1 \cdot 10^{-01}$
$1/5 \cdot 2^{-2}$	$7.1 \cdot 10^{-02}$	0.960	$-1.2 \cdot 10^{-01}$
$1/5 \cdot 2^{-3}$	$3.6 \cdot 10^{-02}$	0.980	$-1.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-4}$	$1.8 \cdot 10^{-03}$	0.990	$-1.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-5}$	$9.3 \cdot 10^{-03}$	0.995	$-1.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-6}$	$4.7 \cdot 10^{-03}$	0.997	$-1.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-7}$	$2.3 \cdot 10^{-04}$	0.998	$-1.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-8}$	$1.1 \cdot 10^{-04}$	0.999	$-1.5 \cdot 10^{-01}$
$1/5 \cdot 2^{-9}$	$5.8 \cdot 10^{-04}$	0.999	$-1.5 \cdot 10^{-01}$

Collocation methods

Consider the following IVP:

$$z'(t) - \frac{M}{t} z(t) = f(t), \quad t \in (0, 1], \quad z(0) = 0.$$

Collocation methods

Consider the following IVP:

$$z'(t) - \frac{M}{t} z(t) = f(t), \quad t \in (0, 1], \quad z(0) = 0.$$

Assume: All eigenvalues of M are negative.

Collocation methods

Consider the following IVP:

$$z'(t) - \frac{M}{t} z(t) = f(t), \quad t \in (0,1], \quad z(0) = 0.$$

Assume: All eigenvalues of M are negative.

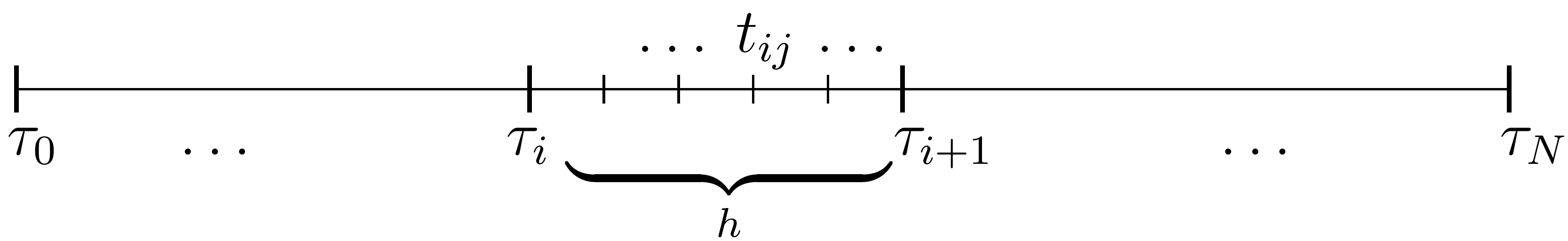
Recall: Let $f \in C[0,1]$. Then a necessary condition for a solution of the problem to be in $C[0,1]$ is

$$Mz(0) = 0 \Leftrightarrow z(0) = 0.$$

Moreover $z \in C^1[0,1]$ and if $f \in C^p[0,1]$, then $z \in C^{p+1}[0,1]$.

Collocation scheme

Consider a mesh of the interval $[0, 1]$.

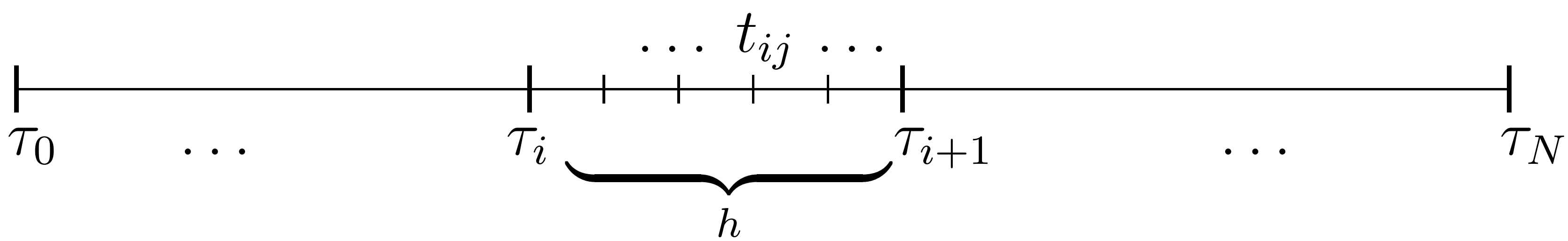


Let

$$\tau_i = ih, \quad i = 0, \dots, N, \;\; h := \frac{1}{N}.$$

Collocation scheme

Consider a mesh of the interval $[0, 1]$.



Let

$$\tau_i = ih, \quad i = 0, \dots, N, \;\; h := \frac{1}{N}.$$

Introduce the grid

$$t_{ij} = \tau_i + \rho_j h, \quad i = 0, \dots, N-1, \quad j = 1, \dots, m,$$

where $0 < \rho_1 < \rho_2 < \cdots < \rho_m < 1$.

Collocation scheme

Let us denote by $\mathcal{P}_m$ the class of piecewise continuous functions on $[0, 1]$ which reduce to a polynomial of degree smaller or equal to m on each subinterval $[\tau_i, \tau_{i+1}]$, $0 \leq i \leq N - 1$.

Collocation scheme

Let us denote by $\mathcal{P}_m$ the class of piecewise continuous functions on $[0,1]$ which reduce to a polynomial of degree smaller or equal to m on each subinterval $[\tau_i, \tau_{i+1}]$, $0 \le i \le N-1$.

Approximate z by a function $p \in \mathcal{P}_m \cap C[0,1]$ satisfying the collocation conditions

$$p'(t_{ij}) - M\frac{p(t_{ij})}{t_{ij}} - f(t_{ij}) = 0, \quad i = 0, \dots, N-1,\ j = 1, \dots, m,$$

and the IC $p(0) = 0$.

Collocation scheme

Let us denote by $\mathcal{P}_m$ the class of piecewise continuous functions on $[0,1]$ which reduce to a polynomial of degree smaller or equal to m on each subinterval $[\tau_i, \tau_{i+1}]$, $0 \le i \le N-1$.

Approximate z by a function $p \in \mathcal{P}_m \cap C[0,1]$ satisfying the collocation conditions

$$p'(t_{ij}) - M\frac{p(t_{ij})}{t_{ij}} - f(t_{ij}) = 0, \quad i = 0, \dots, N-1,\ j = 1, \dots, m,$$

and the IC $p(0) = 0$.

The aim: Show the convergence of the scheme.

Stability

We restrict ourself to the scalar case and consider the function $u \in \mathcal{P}_m \cap C[0,1]$ which satisfies

$$u'(t_{ij}) - \lambda \frac{u(t_{ij})}{t_{ij}} = c_{ij}, \quad u(0) = 0,$$
$$j = 1, \ldots, m, \; i = 0, \ldots, N-1,$$

Stability

We restrict ourself to the scalar case and consider the function $u \in \mathcal{P}_m \cap C[0,1]$ which satisfies

$$u'(t_{ij}) - \lambda \frac{u(t_{ij})}{t_{ij}} = c_{ij}, \quad u(0) = 0,$$
$$j = 1, \ldots, m, \; i = 0, \ldots, N-1,$$

where the c_{ij} are constants. We consider the case $\lambda < 0$.

Stability

We restrict ourself to the scalar case and consider the function $u \in \mathcal{P}_m \cap C[0,1]$ which satisfies

$$u'(t_{ij}) - \lambda \frac{u(t_{ij})}{t_{ij}} = c_{ij}, \quad u(0) = 0,$$
$$j = 1, \dots, m, \; i = 0, \dots, N-1,$$

where the c_{ij} are constants. We consider the case $\lambda < 0$.

Note that

$$tu'(t) - \lambda u(t) - t \sum_{j=1}^{m} l_j \left(\frac{t - \tau_i}{h} \right) c_{ij}$$

is a polynomial of degree $\leq m$, which by the collocation scheme vanishes when $t = t_{ik}$, $k = 1, \dots, m$.

(T2.17-T2.18)

Stability

Again

$$tu'(t) - \lambda u(t) - t\sum_{j=1}^{m} l_j\left(\frac{t-\tau_i}{h}\right)c_{ij}$$

is a polynomial of degree $\leq m$, which by the collocation scheme vanishes when $t = t_{ik}$, $k = 1, \ldots, m$.

Stability

Again

$$tu'(t) - \lambda u(t) - t\sum_{j=1}^{m} l_j\left(\frac{t-\tau_i}{h}\right)c_{ij}$$

is a polynomial of degree $\leq m$, which by the collocation scheme vanishes when $t = t_{ik}$, $k = 1, \ldots, m$.

Hence with a constant ξ_i,

$$tu'(t) - \lambda u(t) - t\sum_{j=1}^{m} l_j\left(\frac{t-\tau_i}{h}\right)c_{ij} = \xi_i w\left(\frac{t-\tau_i}{h}\right),\ t \in [\tau_i, \tau_{i+1}],$$

where $w(s) = (s-\rho_1)(s-\rho_2)\ldots(s-\rho_m)$, $s \in (0,1)$,

and $l_j(s) = w(s)/((s-\rho_j)w'(\rho_j)) \quad j = 1, \ldots, m$.

Stability

For $\lambda < 0$ equation

$$u'(t) - \frac{\lambda}{t} u(t) = \sum_{j=1}^{m} l_j \left(\frac{t - \tau_i}{h} \right) c_{ij} + \frac{1}{t} \xi_i w \left(\frac{t - \tau_i}{h} \right), \; t \in [\tau_i, \tau_{i+1}],$$

has exactly one continuous solution, and this solution is a polynomial of degree $\leq m$,

Stability

For $\lambda < 0$ equation

$$u'(t) - \frac{\lambda}{t} u(t) = \sum_{j=1}^{m} l_j \left(\frac{t - \tau_i}{h} \right) c_{ij} + \frac{1}{t} \xi_i w \left(\frac{t - \tau_i}{h} \right), \; t \in [\tau_i, \tau_{i+1}],$$

has exactly one continuous solution, and this solution is a polynomial of degree $\leq m$,

$$u(t) = t^{\lambda} \sum_{j=1}^{m} \int_0^t s^{-\lambda} l_j \left(\frac{s - \tau_j}{h} \right) ds \, c_{ij} + t^{\lambda} \xi_j \int_0^t s^{-\lambda - 1} w \left(\frac{s - \tau_j}{h} \right) ds.$$

Stability

For $\lambda < 0$ equation

$$u'(t) - \frac{\lambda}{t} u(t) = \sum_{j=1}^{m} l_j \left(\frac{t - \tau_i}{h} \right) c_{ij} + \frac{1}{t} \xi_i w \left(\frac{t - \tau_i}{h} \right), \; t \in [\tau_i, \tau_{i+1}],$$

has exactly one continuous solution, and this solution is a polynomial of degree $\leq m$,

$$u(t) = t^{\lambda} \sum_{j=1}^{m} \int_0^t s^{-\lambda} l_j \left(\frac{s - \tau_j}{h} \right) ds \, c_{ij} + t^{\lambda} \xi_j \int_0^t s^{-\lambda - 1} w \left(\frac{s - \tau_j}{h} \right) ds.$$

On noting that for $i = 0$, $\tau_i = 0$, it follows that

$$\xi_0 = -\lambda \frac{u(0)}{w(0)}.$$

Stability

From the definition of w it follows that

$$w\left(\frac{t-\tau_i}{h}\right) \neq 0, \quad 0 \le t < \tau_i, \quad i = 1, \ldots, N-1.$$

Stability

From the definition of w it follows that

$$w\left(\frac{t-\tau_i}{h}\right) \neq 0, \quad 0 \leq t < \tau_i, \quad i = 1, \ldots, N-1.$$

Then it is possible to eliminate ξ_i by evaluating

$$u(t) = t^{\lambda} \sum_{j=1}^{m} \int_0^t s^{-\lambda} l_j \left(\frac{s-\tau_j}{h}\right) ds\, c_{ij} + t^{\lambda} \xi_j \int_0^t s^{-\lambda-1} w\left(\frac{s-\tau_j}{h}\right) ds.$$

at τ_i and this implies

$$u(t) - A(t,\lambda) u(\tau_i) = \sum_{j=1}^{m} B_j(t,\lambda) c_{ij}, \quad t \in (\tau_i, \tau_{i+1}],\ i = 0, \ldots, N-1.$$

Stability

Theorem: Let all $\lambda(M) < 0$. Then the collocation scheme

Stability

Theorem: Let all $\lambda(M) < 0$. Then the collocation scheme

$$p'(t_{ij}) - M\frac{p(t_{ij})}{t_{ij}} = f(t_{ij}), \quad u(0) = 0,\ j = 1, \ldots, m,\ i = 0, \ldots, N-1$$

has a unique $p \in \mathcal{P}_m \cap C[0,1]$ which satisfies

$$|p(\tau_i + \theta h)| \leq \text{const}\{\tau_{i+1} F_i\}, \quad 0 \leq \theta \leq 1$$

where

$$F_i = \max_{0 \leq l \leq i, 1 \leq j \leq m} |f(t_{lj})|, \quad i = 0, \ldots, N-1.$$

Convergence

Theorem: Let $f \in C^m[0,1]$ and p and z be solutions of the collocation scheme and the analytical problem, subject to the initial conditions

$$p(0) = 0, \quad z(0) = 0,$$

respectively.

Convergence

Theorem: Let $f \in C^m[0,1]$ and p and z be solutions of the collocation scheme and the analytical problem, subject to the initial conditions

$$p(0) = 0, \quad z(0) = 0,$$

respectively.

Then

$$\|p - z\| \le \text{const}\, h^m.$$

Convergence

Theorem: Let $f \in C^m[0,1]$ and p and z be solutions of the collocation scheme and the analytical problem, subject to the initial conditions

$$p(0) = 0, \quad z(0) = 0,$$

respectively.

Then

$$\|p - z\| \leq \text{const}\, h^m.$$

Proof: Let $e \in \mathcal{P}_m \cap C[0,1]$ satisfy

$$e'(t_{ij}) = z'(t_{ij}) - p'(t_{ij}), \quad j = 1, \ldots, m, \quad i = 0, \ldots, N-1,$$
$$e(0) = 0.$$

Convergence

Clearly, for $t \in [\tau_i, \tau_{i+1}]$,

$$e'(t) = \sum_{j=1}^{m} l_j \left(\frac{t - \tau_j}{h} \right) (z'(t_{ij}) - p'(t_{ij})) = \sum_{j=1}^{m} l_j \left(\frac{t - \tau_j}{h} \right) z'(t_{ij}) - p'(t).$$

Convergence

Clearly, for $t \in [\tau_i, \tau_{i+1}]$,

$$e'(t) = \sum_{j=1}^{m} l_j\left(\frac{t-\tau_j}{h}\right)(z'(t_{ij}) - p'(t_{ij})) = \sum_{j=1}^{m} l_j\left(\frac{t-\tau_j}{h}\right) z'(t_{ij}) - p'(t).$$

We know that if $f \in C^m[0,1]$ then $z \in C^{m+1}[0,1]$ and so

$$e'(t) = z'(t) - p'(t) + O(h^m),$$

Convergence

Clearly, for $t \in [\tau_i, \tau_{i+1}]$,

$$e'(t) = \sum_{j=1}^{m} l_j \left(\frac{t - \tau_j}{h} \right) (z'(t_{ij}) - p'(t_{ij})) = \sum_{j=1}^{m} l_j \left(\frac{t - \tau_j}{h} \right) z'(t_{ij}) - p'(t).$$

We know that if $f \in C^m[0,1]$ then $z \in C^{m+1}[0,1]$ and so

$$e'(t) = z'(t) - p'(t) + O(h^m),$$

which by integration yields

$$e(t) = z(t) - p(t) + O(h^m t), \quad e(0) = 0.$$

Convergence

On noting that p and z satisfy the collocation scheme and the analytical problem, respectively, it follows that

$$e'(t_{ij}) - M\frac{e(t_{ij})}{t_{ij}} = O(h^m), \quad j = 1, \ldots, m, \; i = 0, \ldots, N-1, \quad e(0) = 0.$$

Convergence

On noting that p and z satisfy the collocation scheme and the analytical problem, respectively, it follows that

$$e'(t_{ij}) - M\frac{e(t_{ij})}{t_{ij}} = O(h^m), \quad j = 1, \ldots, m, \; i = 0, \ldots, N-1, \quad e(0) = 0.$$

We know, that for

$$p'(t_{ij}) - M\frac{p(t_{ij})}{t_{ij}} = f(t_{ij}), \quad j = 1, \ldots, m, \; i = 0, \ldots, N-1, \quad p(0) = 0,$$

$$\|p\| \leq \|F\| = \max_{i,j} |f(t_{ij})| \Rightarrow \|e\| = O(h^m)$$

and

$$\|p - z\| = \|e\| + O(h^m) = O(h^m).$$

Example

Consider the problem

$$z'(t) = -\frac{1}{t} z(t) + 2\sin t + t\cos t, \qquad t \in (0,1],$$
$$z(0) = 0,$$

with the exact solution $z(t) = t\sin t$.

Example

Numerical experiment with $m = 2$ equidistant points.

Uniform mesh		Error of p at the mesh tau			Error of p at the grid $tcol$		
N	h	**error**	**order**	**err. const.**	**error**	**order**	**err. const.**
10	$1.00e-01$	$3.183e-04$			$3.183e-04$		
20	$5.00e-02$	$7.941e-05$	2.0	$3.205e-02$	$7.941e-05$	2.0	$3.205e-02$
40	$2.50e-02$	$1.984e-05$	2.0	$3.184e-02$	$1.984e-05$	2.0	$3.184e-02$
80	$1.25e-02$	$4.960e-06$	2.0	$3.177e-02$	$4.960e-06$	2.0	$3.177e-02$
160	$6.25e-03$	$1.240e-06$	2.0	$3.175e-02$	$1.240e-06$	2.0	$3.175e-02$
320	$3.13e-03$	$3.100e-07$	2.0	$3.174e-02$	$3.100e-07$	2.0	$3.174e-02$

Example

Numerical experiment with $m = 2$ Gaussian points.

Uniform Mesh		Error of p at the mesh tau			Error of p at the grid $tcol$		
N	h	**error**	**order**	**err. const.**	**error**	**order**	**err. const.**
10	$1.00e-01$	$4.617e-07$			$2.458e-05$		
20	$5.00e-02$	$2.891e-08$	4.0	$4.584e-03$	$3.073e-06$	3.0	$2.458e-02$
40	$2.50e-02$	$1.808e-09$	4.0	$4.616e-03$	$3.840e-07$	3.0	$2.459e-02$
80	$1.25e-02$	$1.130e-10$	4.0	$4.626e-03$	$4.800e-08$	3.0	$2.459e-02$
160	$6.25e-03$	$7.064e-12$	4.0	$4.628e-03$	$6.000e-09$	3.0	$2.458e-02$
320	$3.13e-03$	$4.415e-13$	4.0	$4.628e-03$	$7.500e-10$	3.0	$2.458e-02$

Applications

Complex Ginzburg-Landau equation

$$\mathrm{i}\frac{\partial u}{\partial t} + (1 - \mathrm{i}\varepsilon)\Delta u + (1 + \mathrm{i}\delta)|u|^2 u = 0, \quad t \in (0, \infty), \;\; x \in \mathbb{R}^3$$

Applications

Complex Ginzburg-Landau equation

$$\mathrm{i}\frac{\partial u}{\partial t} + (1 - \mathrm{i}\varepsilon)\Delta u + (1 + \mathrm{i}\delta)|u|^2 u = 0, \quad t \in (0,\infty), \;\; x \in \mathbb{R}^3$$

We are interested in *self-similar blow-up* solutions

Ansatz

$$u(x,t) = L(\tau)y(\tau)$$

$$\tau = \tau(x,t) = \sqrt{\frac{x_1^2 + x_2^2 + x_3^2}{2a(T-t)}}, \quad \lim_{t\to T} L(\tau(x,t)) = \infty$$

Applications

Complex Ginzburg-Landau equation

$$\mathrm{i}\frac{\partial u}{\partial t} + (1 - \mathrm{i}\varepsilon)\Delta u + (1 + \mathrm{i}\delta)|u|^2 u = 0, \quad t \in (0, \infty), \ \ x \in \mathbb{R}^3$$

We are interested in *self-similar blow-up* solutions

Ansatz

$$u(x,t) = L(\tau)y(\tau)$$

$$\tau = \tau(x,t) = \sqrt{\frac{x_1^2 + x_2^2 + x_3^2}{2a(T-t)}}, \quad \lim_{t \to T} L(\tau(x,t)) = \infty$$

Nonlinear optics, models of turbulence, superconductivity

Budd, Koch, W. (2006)

Similarity reduction

Find $y : [0, \infty) \to \mathbb{C}$

$$(1-\mathrm{i}\varepsilon)\left(y''(\tau) + \frac{2}{\tau}y'(\tau)\right) - y(\tau) + \mathrm{i}a(\tau y(\tau))' + (1+\mathrm{i}\delta)|y(\tau)|^2 y(\tau) = 0$$

$$y'(0) = 0, \quad \Im y(0) = 0, \quad \lim_{\tau\to\infty} \tau y'(\tau) = 0$$

Similarity reduction

Find $y : [0, \infty) \to \mathbb{C}$

$$(1-\mathrm{i}\varepsilon)\left(y''(\tau) + \frac{2}{\tau}y'(\tau)\right) - y(\tau) + \mathrm{i}a(\tau y(\tau))' + (1+\mathrm{i}\delta)|y(\tau)|^2 y(\tau) = 0$$

$$y'(0) = 0, \quad \Im y(0) = 0, \quad \lim_{\tau\to\infty} \tau y'(\tau) = 0$$

Implicit, parameter dependent ODE with singular points and unknown parameter a posed on a semi-infinite interval

Complex Ginzburg-Landau equation

Pathfollowing for CGL:

Find $y : [0, \infty) \to \mathbb{C}$

$$(1-\mathrm{i}\varepsilon)\left(y''(\tau)+\frac{2}{\tau}y'(\tau)\right)-y(\tau)+\mathrm{i}a(\tau y(\tau))'+(1+\mathrm{i}\delta)|y(\tau)|^2 y(\tau)=0$$

$$y'(0)=0, \quad \Im y(0)=0, \quad \lim_{\tau\to\infty} \tau y'(\tau)=0$$

a . . . unknown, nonnegative parameter

Complex Ginzburg-Landau equation

Pathfollowing for CGL:

Find $y : [0, \infty) \to \mathbb{C}$

$$(1-\mathrm{i}\varepsilon)\left(y''(\tau) + \frac{2}{\tau}y'(\tau)\right) - y(\tau) + \mathrm{i}a(\tau y(\tau))' + (1+\mathrm{i}\delta)|y(\tau)|^2 y(\tau) = 0$$

$$y'(0) = 0, \quad \Im y(0) = 0, \quad \lim_{\tau\to\infty} \tau y'(\tau) = 0$$

a . . . unknown, nonnegative parameter

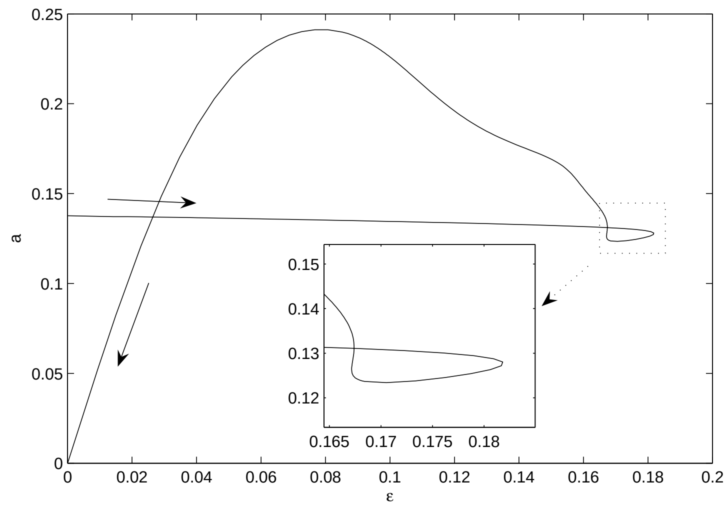
$\varepsilon = \delta = 0$: Nonlinear Schrödinger equation (NLS)

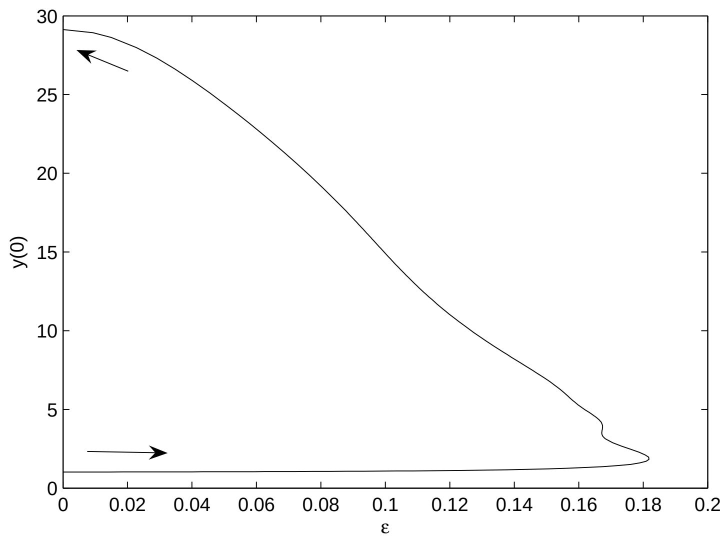
Pathfollowing for CGL

Set $\delta = 0$, vary ε:

Pathfollowing for CGL

Set $\delta = 0$, vary ε:

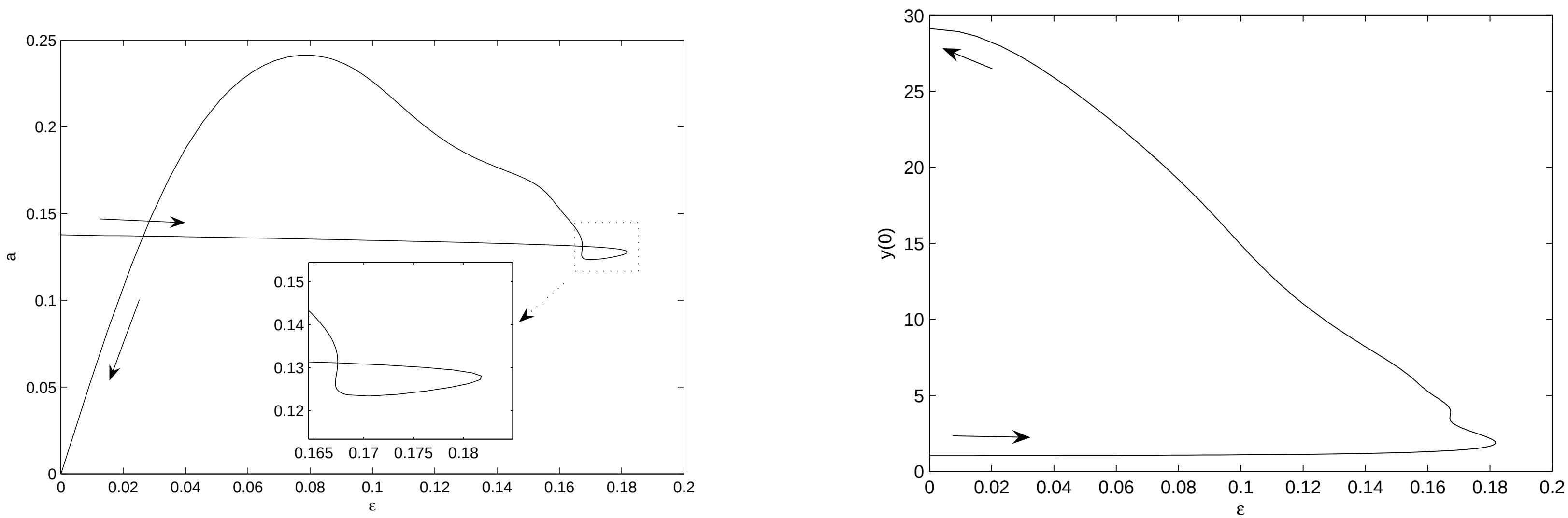




Values of a (left) and $y(0)$ (right) along a branch bifurcating from the NLS

Pathfollowing for CGL

Set $\delta = 0$, vary ε:



Values of a (left) and $y(0)$ (right) along a branch bifurcating from the NLS

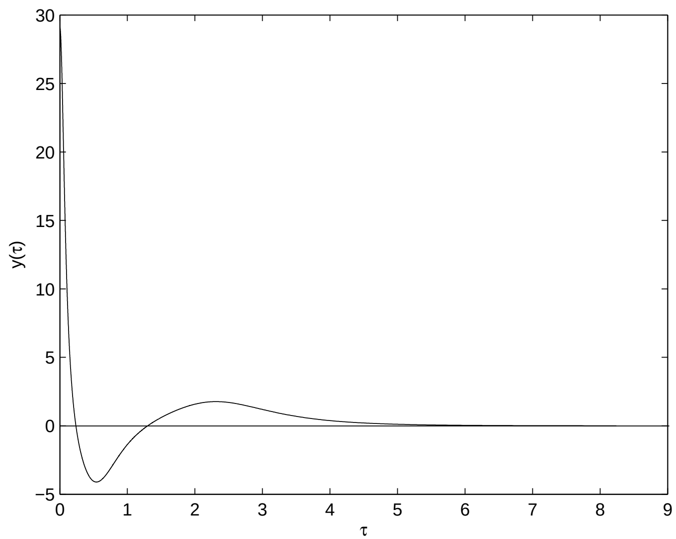
Γ has three turning points, but no bifurcation points

Pathfollowing for CGL

Path Γ terminates at a real-valued, multi-bump solution of the NLS

Pathfollowing for CGL

Path Γ terminates at a real-valued, multi-bump solution of the NLS



Real solution of the NLS computed by pathfollowing

Density profile equation in hydrodynamics

Bubbles: Density profile equation in hydrodynamics

Kitzhofer, Koch, Lima, W. (2005)

$$\rho''(r) + \frac{N-1}{r}\rho'(r) = 4(\rho(r)+1)\rho(r)(\rho(r)-\xi), \quad r \in (0,\infty)$$

Reformulation for $s \in (0,1]$, $y_1(s) = \rho(s)$, $y_2(s) = \rho(1/s)$:

$$y_1''(s) + \frac{N-1}{s}y_1'(s) = 4(y_1(s)+1)y_1(s)(y_1(s)-\xi)$$

$$y_2''(s) - \frac{N-3}{s}y_2'(s) = 4\frac{1}{s^4}(y_2(s)+1)y_2(s)(y_2(s)-\xi)$$

Density profile equation in hydrodynamics

Bubbles: Density profile equation in hydrodynamics

Kitzhofer, Koch, Lima, W. (2005)

$$\rho''(r) + \frac{N-1}{r}\rho'(r) = 4(\rho(r)+1)\rho(r)(\rho(r)-\xi), \quad r \in (0,\infty)$$

Reformulation for $s \in (0,1]$, $y_1(s) = \rho(s)$, $y_2(s) = \rho(1/s)$:

$$y_1''(s) + \frac{N-1}{s}y_1'(s) = 4(y_1(s)+1)y_1(s)(y_1(s)-\xi)$$
$$y_2''(s) - \frac{N-3}{s}y_2'(s) = 4\frac{1}{s^4}(y_2(s)+1)y_2(s)(y_2(s)-\xi)$$

Well-posedness ✓

Implicit formulation: Optimal conditioning $cond = O(h^{-2})$

Density profile equation in hydrodynamics

Bubbles: Density profile equation in hydrodynamics

Kitzhofer, Koch, Lima, W. (2005)

$$\rho''(r) + \frac{N-1}{r}\rho'(r) = 4(\rho(r)+1)\rho(r)(\rho(r)-\xi), \quad r \in (0,\infty)$$

Reformulation for $s \in (0,1]$, $y_1(s) = \rho(s)$, $y_2(s) = \rho(1/s)$:

$$y_1''(s) + \frac{N-1}{s}y_1'(s) = 4(y_1(s)+1)y_1(s)(y_1(s)-\xi)$$

$$y_2''(s) - \frac{N-3}{s}y_2'(s) = 4\frac{1}{s^4}(y_2(s)+1)y_2(s)(y_2(s)-\xi)$$

Well-posedness $\checkmark$

Implicit formulation: Optimal conditioning $cond = O(h^{-2})$

Start animation Bubble!

Reaction-diffusion systems

Staněk, Pulverer, W. (2008)

$$u''(t) = \lambda g(u(t)) = \frac{\lambda}{\sqrt{u(t)}}, \quad t \in [0,1]$$

$$u(0) = 1, \quad u(1) = 1$$

$\lambda \geq 0 \ldots$ parameter, u *may vanish*

Reaction-diffusion systems

Staněk, Pulverer, W. (2008)

$$u''(t) = \lambda g(u(t)) = \frac{\lambda}{\sqrt{u(t)}}, \;\; t \in [0,1]$$

$$u(0) = 1, \quad u(1) = 1$$

$\lambda \geq 0 \ldots$ parameter, u *may vanish*

Application: Steady-state of reaction-diffusion of several chemical species

Reaction-diffusion systems

Staněk, Pulverer, W. (2008)

$$u''(t) = \lambda g(u(t)) = \frac{\lambda}{\sqrt{u(t)}}, \quad t \in [0,1]$$

$$u(0) = 1, \quad u(1) = 1$$

$\lambda \geq 0 \ldots$ parameter, u *may vanish*

Application: Steady-state of reaction-diffusion of several chemical species

Problem: Characterize the relation between the values of λ and number and type of solutions, *positive, pseudo dead core, and dead core!*

Dead core solutions

Original formulation

$$u''(t) = \frac{\lambda}{\sqrt{u(t)}}, \quad u(0) = 1, \;\; u(1) = 1$$

Dead core solutions

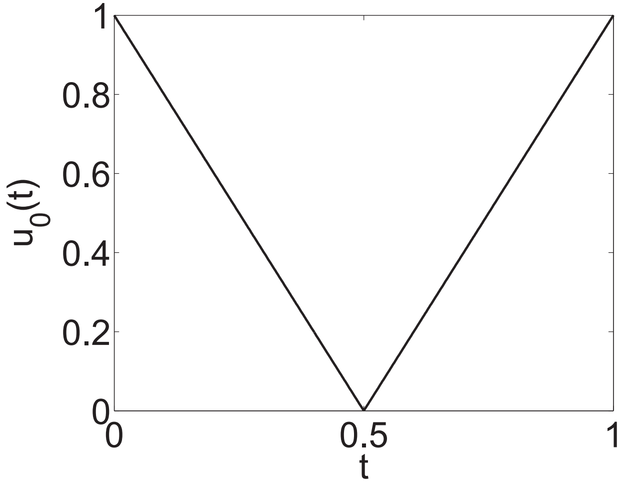
Original formulation

$$u''(t) = \frac{\lambda}{\sqrt{u(t)}}, \quad u(0) = 1, \;\; u(1) = 1$$

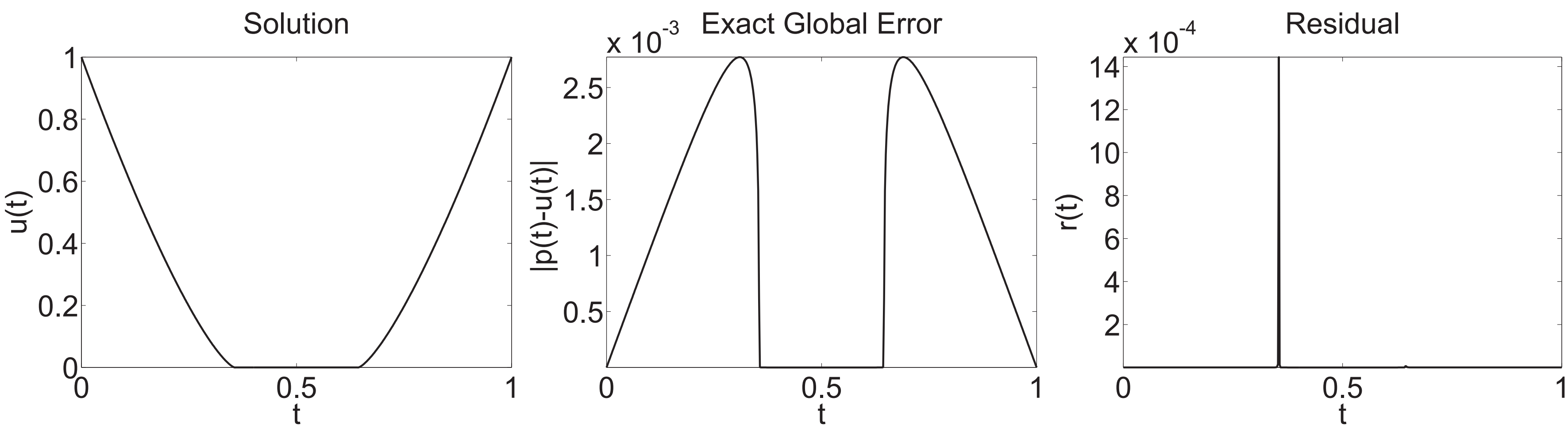
New formulation

$$u''(t)\sqrt{u(t)}u(t) = \lambda u(t), \quad u(0) = 1, \;\; u(1) = 1$$

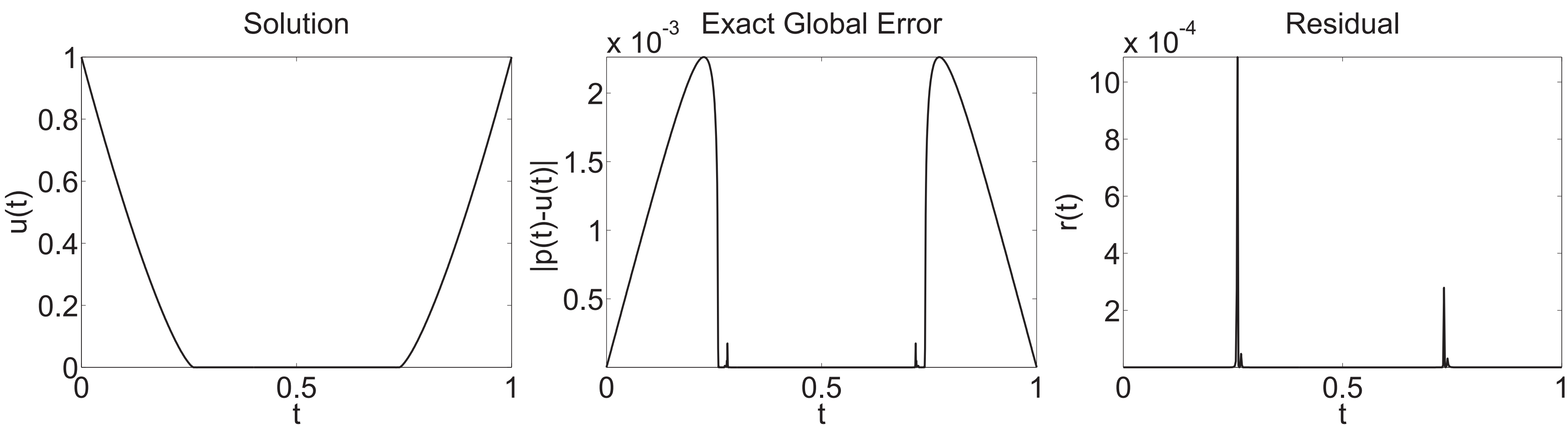
Initial profile



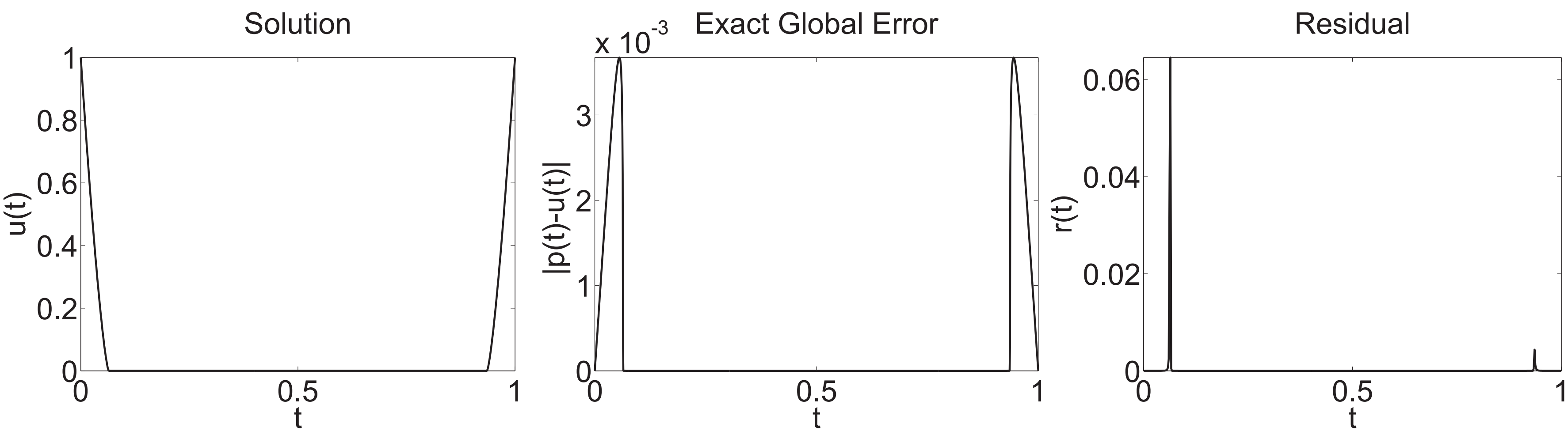
Dead core solution, $\lambda = \frac{32}{9}$



Dead core solution, $\lambda = \frac{60}{9}$



Dead core solution, $\lambda = \frac{1000}{9}$



Summary

We discussed convergence of two numerical schemes applied to

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0, 1], \quad z(0) = 0, \quad \lambda(M) < 0.$$

Summary

We discussed convergence of two numerical schemes applied to

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0,1], \quad z(0) = 0, \quad \lambda(M) < 0.$$

- For $z \in C^2[0,1]$, forward Euler method shows classical convergence order one.

Summary

We discussed convergence of two numerical schemes applied to

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0,1], \quad z(0) = 0, \quad \lambda(M) < 0.$$

- For $z \in C^2[0,1]$, forward Euler method shows classical convergence order one.
- For $z \in C^{m+1}[0,1]$, collocation at m points shows the stage order m uniformly in t.

Summary

We discussed convergence of two numerical schemes applied to

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0,1], \quad z(0) = 0, \quad \lambda(M) < 0.$$

- For $z \in C^2[0,1]$, forward Euler method shows classical convergence order one.
- For $z \in C^{m+1}[0,1]$, collocation at m points shows the stage order m uniformly in t.
- Superconvergence cannot be guaranteed in general.

Summary

We discussed convergence of two numerical schemes applied to

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0,1], \quad z(0) = 0, \quad \lambda(M) < 0.$$

- For $z \in C^2[0,1]$, forward Euler method shows classical convergence order one.
- For $z \in C^{m+1}[0,1]$, collocation at m points shows the stage order m uniformly in t.
- Superconvergence cannot be guaranteed in general.
- For stability we invert the discrete operator.

Summary

We discussed convergence of two numerical schemes applied to

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0,1], \quad z(0) = 0, \quad \lambda(M) < 0.$$

- For $z \in C^2[0,1]$, forward Euler method shows classical convergence order one.
- For $z \in C^{m+1}[0,1]$, collocation at m points shows the stage order m uniformly in t.
- Superconvergence cannot be guaranteed in general.
- For stability we invert the discrete operator.
- Open question: Convergence of collocation for the singularity of second kind.