INVESTMENTS IN EDUCATION DEVELOPMENT

Streamlining the Mathematics Studies at the Faculty of Science of Palacky University in Olomouc

Registry number: CZ.1.07/2.2.00/28.0141

Numerical Treatment of Singular BVPs in ODEs

Ewa Weinmüller

Analysis and Scientific Computing, Vienna University of Technology

Palacky University, Olomouc, Czech Republic, 23 – 25 June 2014

Contents

Basic concepts – work by H. Keller ✓

Motivation and introductory remarks

Forward Euler method

- *Stability*
- *Consistency and convergence*
- *Numerical examples*

Collocation method

- *Existence and uniqueness of the collocation solution*
- *Convergence*
- *Numerical example*

Motivation

Complex Ginzburg-Landau equation

$$\mathrm{i}\frac{\partial u}{\partial t} + (1 - \mathrm{i}\varepsilon)\Delta u + (1 + \mathrm{i}\delta)|u|^2 u = 0, \quad t \in (0, \infty), \;\; x \in \mathbb{R}^3$$

Motivation

Complex Ginzburg-Landau equation

$$\mathrm{i}\frac{\partial u}{\partial t} + (1 - \mathrm{i}\varepsilon)\Delta u + (1 + \mathrm{i}\delta)|u|^2 u = 0, \quad t \in (0, \infty), \ \ x \in \mathbb{R}^3$$

We are interested in *self-similar blow-up* solutions

Ansatz

$$u(x,t) = L(\tau)y(\tau)$$

$$\tau = \tau(x,t) = \sqrt{\frac{x_1^2 + x_2^2 + x_3^2}{2a(T-t)}}, \quad \lim_{t \to T} L(\tau(x,t)) = \infty$$

Motivation

Complex Ginzburg-Landau equation

$$\mathrm{i}\frac{\partial u}{\partial t} + (1 - \mathrm{i}\varepsilon)\Delta u + (1 + \mathrm{i}\delta)|u|^2 u = 0, \quad t \in (0, \infty), \;\; x \in \mathbb{R}^3$$

We are interested in *self-similar blow-up* solutions

Ansatz

$$u(x,t) = L(\tau)y(\tau)$$

$$\tau = \tau(x,t) = \sqrt{\frac{x_1^2 + x_2^2 + x_3^2}{2a(T-t)}}, \quad \lim_{t \to T} L(\tau(x,t)) = \infty$$

Nonlinear optics, models of turbulence, superconductivity

Budd, Koch, W. (2006)

Similarity reduction

Find $y : [0, \infty) \to \mathbb{C}$

$$(1-\mathrm{i}\varepsilon)\left(y''(\tau) + \frac{2}{\tau}y'(\tau)\right) - y(\tau) + \mathrm{i}a(\tau y(\tau))' + (1+\mathrm{i}\delta)|y(\tau)|^2 y(\tau) = 0$$

$$y'(0) = 0, \quad \Im y(0) = 0, \quad \lim_{\tau\to\infty} \tau y'(\tau) = 0$$

Similarity reduction

Find $y : [0, \infty) \to \mathbb{C}$

$$(1 - \mathrm{i}\varepsilon)\left(y''(\tau) + \frac{2}{\tau} y'(\tau)\right) - y(\tau) + \mathrm{i}a(\tau y(\tau))' + (1 + \mathrm{i}\delta)|y(\tau)|^2 y(\tau) = 0$$

$$y'(0) = 0, \quad \Im y(0) = 0, \quad \lim_{\tau \to \infty} \tau y'(\tau) = 0$$

Implicit, parameter dependent ODE with singular points and unknown parameter a posed on a semi-infinite interval

Similarity reduction

Find $y : [0, \infty) \to \mathbb{C}$

$$(1-\mathrm{i}\varepsilon)\left(y''(\tau) + \frac{2}{\tau}y'(\tau)\right) - y(\tau) + \mathrm{i}a(\tau y(\tau))' + (1+\mathrm{i}\delta)|y(\tau)|^2 y(\tau) = 0$$

$$y'(0) = 0, \quad \Im y(0) = 0, \quad \lim_{\tau\to\infty} \tau y'(\tau) = 0$$

Implicit, parameter dependent ODE with singular points and unknown parameter a posed on a semi-infinite interval

Need: Code equipped with error estimation for the global error and grid adaptation strategy

Introduction: Analytical properties

$$z'(t) = \frac{1}{t^\alpha} f(t, z(t)), \quad t \in (0, 1], \quad \alpha \geq 1$$

$$z'(\tau) = \tau^\beta f(\tau, z(\tau)), \quad \tau \in [1, \infty), \quad \beta \geq -1$$

Introduction: Analytical properties

$$z'(t) = \frac{1}{t^\alpha} f(t, z(t)), \quad t \in (0, 1], \quad \alpha \geq 1$$

$$z'(\tau) = \tau^\beta f(\tau, z(\tau)), \quad \tau \in [1, \infty), \quad \beta \geq -1$$

Problems posed on a *semi-infinite* interval are transformed to a *finite* interval by $t := 1/\tau$

$$z'(t) = -\frac{1}{t^{\beta+2}} f(1/t, z(1/t)), \quad t \in (0, 1], \quad \beta + 2 \geq 1$$

Introduction: Analytical properties

$$z'(t) = \frac{1}{t^\alpha} f(t, z(t)), \quad t \in (0, 1], \quad \alpha \geq 1$$

$$z'(\tau) = \tau^\beta f(\tau, z(\tau)), \quad \tau \in [1, \infty), \quad \beta \geq -1$$

Problems posed on a *semi-infinite* interval are transformed to a *finite* interval by $t := 1/\tau$

$$z'(t) = -\frac{1}{t^{\beta+2}} f(1/t, z(1/t)), \quad t \in (0, 1], \quad \beta + 2 \geq 1$$

Second order systems

$$z''(t) = \frac{1}{t^\alpha} f(t, z'(t)) + \frac{1}{t^{\alpha+1}} g(t, z(t)), \quad t \in (0, 1], \quad \alpha \geq 1$$

Introduction: Smooth solutions

Model problem $z'(t) = \underbrace{\frac{M(t)}{t} z(t)}_{\frac{M(0)+tC(t)}{t} z(t)} + f(t, z(t)), \quad t \in (0, 1]$

Introduction: Smooth solutions

Model problem $z'(t) = \underbrace{\frac{M(t)}{t} z(t)}_{\frac{M(0)+tC(t)}{t} z(t)} + f(t, z(t)), \quad t \in (0,1]$

Interested in $z \in C[0,1]$, even $z \in C^m[0,1],\ m \geq 1$

Introduction: Smooth solutions

Model problem $$z'(t) = \underbrace{\frac{M(t)}{t} z(t)}_{\frac{M(0)+tC(t)}{t} z(t)} + f(t, z(t)), \quad t \in (0,1]$$

Interested in $z \in C[0,1]$, even $z \in C^m[0,1]$, $m \geq 1$

Direction field unsmooth and therefore

unbounded solutions $z \in C(0,1]$ exist

Introduction: Smooth solutions

Model problem $$z'(t) = \underbrace{\frac{M(t)}{t} z(t)}_{\frac{M(0)+tC(t)}{t} z(t)} + f(t, z(t)), \quad t \in (0,1]$$

Interested in $z \in C[0,1]$, even $z \in C^m[0,1]$, $m \geq 1$

Direction field unsmooth and therefore

unbounded solutions $z \in C(0,1]$ exist

Extend $z \in C(0,1]$ to $z \in C[0,1] \quad \Leftrightarrow \quad M(0)z(0) = 0$

Introduction: Smooth solutions

Model problem $z'(t) = \underbrace{\frac{M(t)}{t} z(t)}_{\frac{M(0)+tC(t)}{t} z(t)} + f(t, z(t)), \quad t \in (0, 1]$

Interested in $z \in C[0, 1]$, even $z \in C^m[0, 1]$, $m \geq 1$

Direction field unsmooth and therefore

unbounded solutions $z \in C(0, 1]$ exist

Extend $z \in C(0, 1]$ to $z \in C[0, 1] \quad \Leftrightarrow \quad M(0)z(0) = 0$

Smoothness of z depends on smoothness of f and on the size of the positive real parts of the eigenvalues of $M(0)$

de Hoog, Weiss (1976), de Hoog, Weiss (1980), W. (1980)

Introduction: Numerical approach

Main assumption: Analytical problem is

well-posed with a locally unique smooth solution

Introduction: Numerical approach

Main assumption: Analytical problem is

well-posed with a locally unique smooth solution

- Numerical method: Robust with respect to singularity

$$\|\text{global error}\| = O(h^m), \ m \text{ reasonably large}$$

Our choice is *polynomial collocation*

Introduction: Numerical approach

Main assumption: Analytical problem is

well-posed with a locally unique smooth solution

- Numerical method: Robust with respect to singularity

  $$\|\text{global error}\| = O(h^m),\ m \text{ reasonably large}$$

  Our choice is *polynomial collocation*

- Error estimation: Robust and asymptotically correct

  $$\|\text{global error - error estimate}\| = O(h^{m+\gamma}),\ \gamma > 0$$

  Our choice is $h - h/2$ *strategy*

Introduction: Numerical approach

Main assumption: Analytical problem is

well-posed with a locally unique smooth solution

- Numerical method: Robust with respect to singularity

  $\|\text{global error}\| = O(h^m), \; m$ reasonably large

  Our choice is *polynomial collocation*

- Error estimation: Robust and asymptotically correct

  $\|\text{global error - error estimate}\| = O(h^{m+\gamma}), \; \gamma > 0$

  Our choice is $h - h/2$ *strategy*

- Adaptive mesh selection:

  Meshes unaffected by the unsmooth (!) direction field

MATLAB code `bvpsuite` – Scope

Kitzhofer, Koch, Pulverer, Simon, W. (2009)

http://www.math.tuwien.ac.at/~ewa/

Implicit nonlinear mixed order system of ODEs

$$F\Big(t, p_1, \ldots, p_s, z_1(t), z_1'(t), \ldots, z_1^{(l_1)}(t), \ldots$$

$$z_n(t), z_n'(t), \ldots, z_n^{(l_n)}(t)\Big) = 0$$

$$B\Big(p_1, \ldots, p_s, z_1(c_1), \ldots, z_1^{(l_1-1)}(c_1), \ldots, z_n(c_1), \ldots$$

$$z_n(c_q), \ldots, z_n^{(l_n-1)}(c_q)\Big) = 0$$

MATLAB code `bvpsuite` – Scope

Kitzhofer, Koch, Pulverer, Simon, W. (2009)

http://www.math.tuwien.ac.at/˜ewa/

Implicit nonlinear mixed order system of ODEs

$$F\Big(t, p_1, \dots, p_s, z_1(t), z_1'(t), \dots, z_1^{(l_1)}(t), \dots$$

$$z_n(t), z_n'(t), \dots, z_n^{(l_n)}(t)\Big) = 0$$

$$B\Big(p_1, \dots, p_s, z_1(c_1), \dots, z_1^{(l_1-1)}(c_1), \dots, z_n(c_1), \dots$$

$$z_n(c_q), \dots, z_n^{(l_n-1)}(c_q)\Big) = 0$$

Unknowns: $z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T$, p_i, $i = 1, \dots, s$

In general, $t \in [a, b]$ or $t \in [a, \infty)$, $a \geq 0$

Main features

- Implicit, mixed order equations – specification of the highest derivatives of $z_i(t)$, $i = 1, \ldots, n$ is necessary

Main features

- Implicit, mixed order equations – specification of the highest derivatives of $z_i(t)$, $i = 1, \ldots, n$ is necessary
- Order of the method, *standard* $m = 2$ *to* $m = 8$, is chosen automatically depending on TOL_a

Main features

- Implicit, mixed order equations – specification of the highest derivatives of $z_i(t)$, $i = 1, \ldots, n$ is necessary
- Order of the method, *standard* $m = 2$ *to* $m = 8$, is chosen automatically depending on TOL_a
- We implemented *Gaussian points (standard)*, Lobatto, and uniform. User's own specification is possible

Main features

- Implicit, mixed order equations – specification of the highest derivatives of $z_i(t)$, $i = 1, \ldots, n$ is necessary
- Order of the method, *standard* $m = 2$ *to* $m = 8$, is chosen automatically depending on TOL_a
- We implemented *Gaussian points (standard)*, Lobatto, and uniform. User's own specification is possible
- Specification of different TOL_a and TOL_r parameters for different solution components is possible

Main features

- Implicit, mixed order equations – specification of the highest derivatives of $z_i(t)$, $i = 1, \ldots, n$ is necessary
- Order of the method, *standard* $m = 2$ *to* $m = 8$, is chosen automatically depending on TOL_a
- We implemented *Gaussian points (standard)*, Lobatto, and uniform. User's own specification is possible
- Specification of different TOL_a and TOL_r parameters for different solution components is possible
- *Mesh adaptation is the standard choice* but fixed grid is also possible

Main features

- Implicit, mixed order equations – specification of the highest derivatives of $z_i(t),\ i = 1, \dots, n$ is necessary
- Order of the method, *standard* $m = 2$ *to* $m = 8$, is chosen automatically depending on TOL_a
- We implemented *Gaussian points (standard)*, Lobatto, and uniform. User's own specification is possible
- Specification of different TOL_a and TOL_r parameters for different solution components is possible
- *Mesh adaptation is the standard choice* but fixed grid is also possible
- Driver routine for the interval reduction $[a, \infty),\ a \geq 0$ *to a finite domain* in case of *Dirichlet BC at infinity* is available

Introduction: Additional modules

- Pathfollowing for parameter-dependent problems–turning points covered.

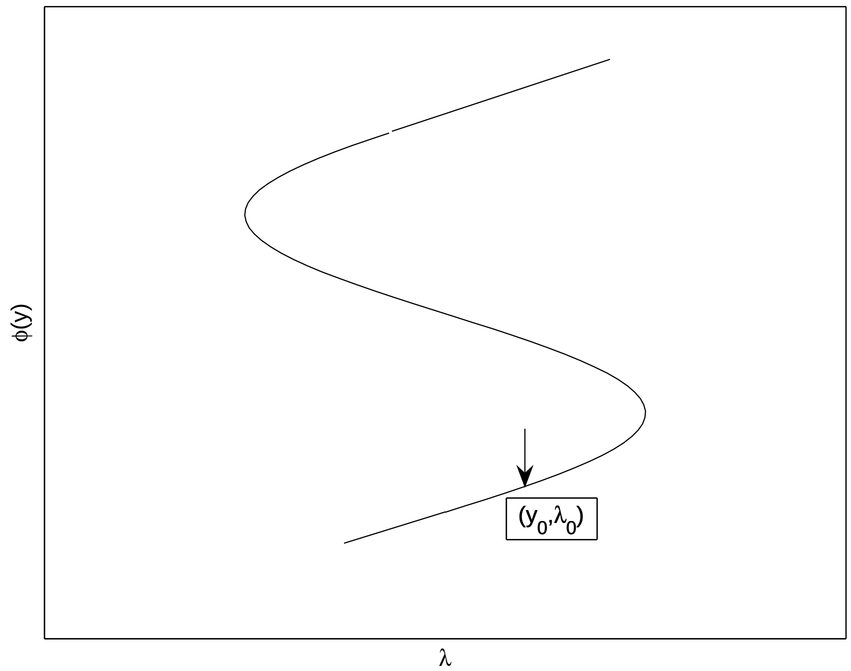
Introduction: Additional modules

- Pathfollowing for parameter-dependent problems–turning points covered.
- Eigenvalue problems–Sturm-Liouville problems with singularities covered.

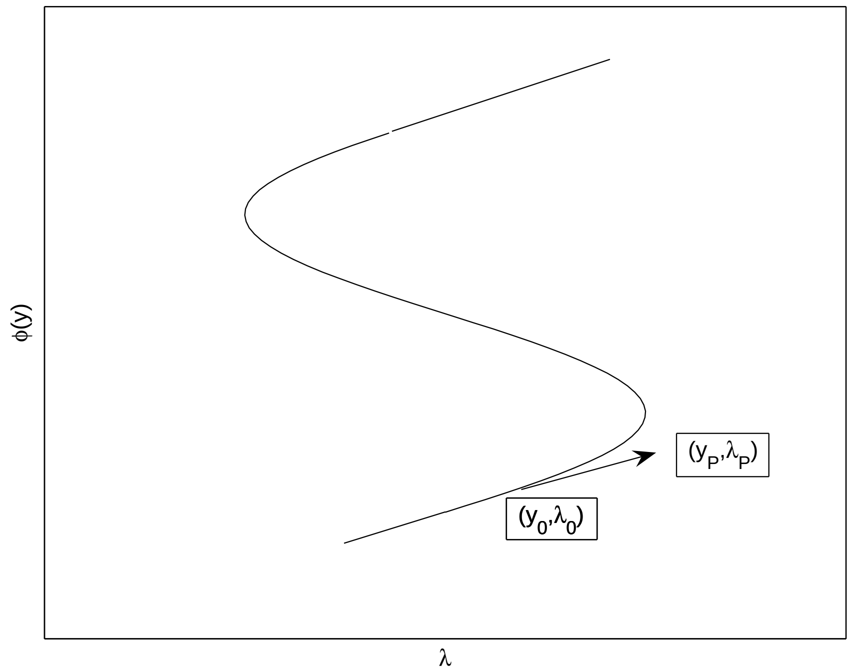
Introduction: Additional modules

- Pathfollowing for parameter-dependent problems–turning points covered.
- Eigenvalue problems–Sturm-Liouville problems with singularities covered.
- Problems posed on semi-infinite intervals, $t \in [a, \infty], a \leq 0$.

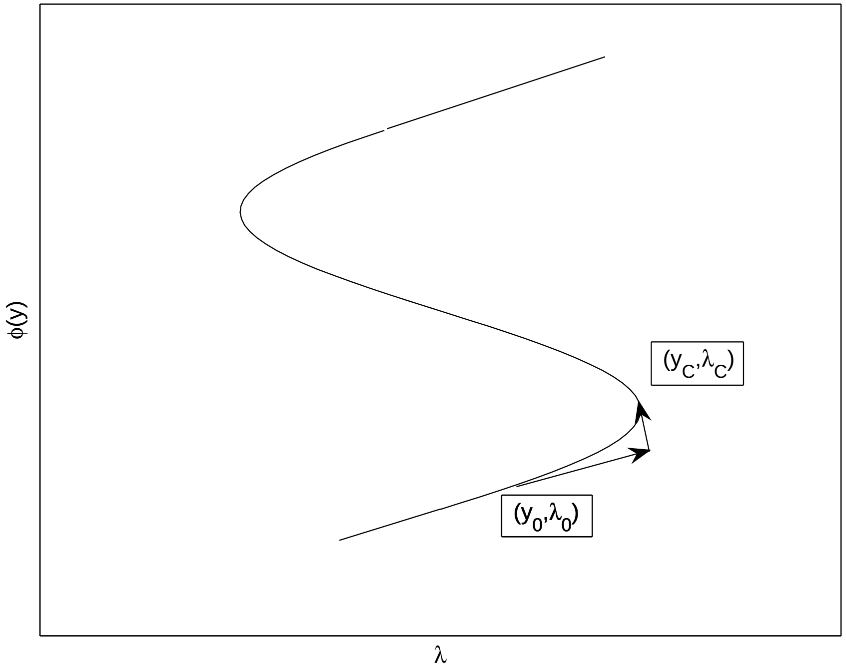
Introduction: Pseudo arclength parametrization



Introduction: Pseudo arclength parametrization



Introduction: Pseudo arclength parametrization



Introduction: case study

Case study: Consider linear system

$$z'(t) - \frac{D}{t} z(t) = f(t), \ \ t \in (0,1], \ \ D = \text{diag}(-1,2)$$

$$z_1(0) = 0, \ \ z_2(1) = \beta$$

Introduction: case study

Case study: Consider linear system

$$z'(t) - \frac{D}{t} z(t) = f(t), \;\; t \in (0,1], \;\; D = \text{diag}(-1, 2)$$

$$z_1(0) = 0, \;\; z_2(1) = \beta$$

Box scheme on equidistant mesh, $h = 1/N$ (T2.1-T2.2):

$$\frac{\xi_i - \xi_{i-1}}{h} - \frac{D}{t_{i-1/2}} \frac{\xi_i + \xi_{i-1}}{2} = f_{i-1/2}, \; i = 1, \ldots, N$$

$$\xi_{1,0} = 0, \; \xi_{2,N} = \beta$$

Introduction: case study

Case study: Consider linear system

$$z'(t) - \frac{D}{t} z(t) = f(t), \quad t \in (0,1], \quad D = \text{diag}(-1, 2)$$

$$z_1(0) = 0, \quad z_2(1) = \beta$$

Box scheme on equidistant mesh, $h = 1/N$ (T2.1-T2.2):

$$\frac{\xi_i - \xi_{i-1}}{h} - \frac{D}{t_{i-1/2}} \frac{\xi_i + \xi_{i-1}}{2} = f_{i-1/2}, \; i = 1, \ldots, N$$

$$\xi_{1,0} = 0, \; \xi_{2,N} = \beta$$

Question: convergence $\lim_{h \to 0} \|\xi_h - z_{\Delta_h}\| = 0$?

$$\xi_h := (\xi_0, \ldots, \xi_i, \ldots, \xi_N) \qquad z_{\Delta_h} := (z(0), \ldots, z(t_i), \ldots, z(1))$$

Difficulties

Classical approach for regular case $z'(t) - Dz(t) = f(t)$

Difficulties

Classical approach for regular case $z'(t) - Dz(t) = f(t)$

Stability + Consistency = Convergence!

Difficulties

Classical approach for regular case $z'(t) - Dz(t) = f(t)$

Stability + Consistency = Convergence!

- Write the scheme as $\Phi_h \xi_h = f_h, \;\; \Phi_h : \mathbb{R}^{2(N+1)} \to \mathbb{R}^{2(N+1)}$.

Difficulties

Classical approach for regular case $z'(t) - Dz(t) = f(t)$

Stability + Consistency = Convergence!

- Write the scheme as $\Phi_h \xi_h = f_h, \ \ \Phi_h : \mathbb{R}^{2(N+1)} \to \mathbb{R}^{2(N+1)}$.
- Show that a solution of $\Phi_h \xi_h = f_h$ exists.

Difficulties

Classical approach for regular case $z'(t) - Dz(t) = f(t)$

Stability + Consistency = Convergence!

- Write the scheme as $\Phi_h \xi_h = f_h, \;\; \Phi_h : \mathbb{R}^{2(N+1)} \to \mathbb{R}^{2(N+1)}$.
- Show that a solution of $\Phi_h \xi_h = f_h$ exists.
- Show stability (T2.3)

$$\|\xi_h - \eta_h\| \leq S \, \|\Phi_h \xi_h - \Phi_h \eta_h\|.$$

Difficulties

Classical approach for regular case $z'(t) - Dz(t) = f(t)$

Stability + Consistency = Convergence!

- Write the scheme as $\Phi_h \xi_h = f_h, \;\; \Phi_h : \mathbb{R}^{2(N+1)} \to \mathbb{R}^{2(N+1)}$.
- Show that a solution of $\Phi_h \xi_h = f_h$ exists.
- Show stability (T2.3)

$$\|\xi_h - \eta_h\| \leq S \, \|\Phi_h \xi_h - \Phi_h \eta_h\|.$$

- Show consistency for z sufficiently smooth (T2.4)

$$(\Phi_h z_{\Delta_h})_i = \frac{z(t_i) - z(t_{i-1})}{h} - D \frac{z(t_i) + z(t_{i-1})}{2} = f_{i-1/2} + O(h^2).$$

Difficulties

Consider $\Phi_h \xi_h = f_h, \;\; \Phi_h z_{\Delta_h} = f_h + O(h^2)$

Difficulties

Consider $\Phi_h \xi_h = f_h, \;\; \Phi_h z_{\Delta_h} = f_h + O(h^2)$

Then $\|\xi_h - z_{\Delta_h}\| \le S\,\|\Phi_h \xi_h - \Phi_h z_{\Delta_h}\| \le S\,h^2$

Difficulties

Consider $\Phi_h \xi_h = f_h, \ \Phi_h z_{\Delta_h} = f_h + O(h^2)$

Then $\|\xi_h - z_{\Delta_h}\| \le S \, \|\Phi_h \xi_h - \Phi_h z_{\Delta_h}\| \le S \, h^2$

- Difficulty in singular case

$$(\Phi_h z_{\Delta_h})_i = \frac{z(t_i) - z(t_{i-1})}{h} - \frac{D}{t_{i-1/2}} \frac{z(t_i) + z(t_{i-1})}{2} = f_{i-1/2} + O\left(\frac{h^2}{t_{i-1/2}}\right)$$

$$\Phi_h \xi_h = f_h, \ \ \Phi_h z_{\Delta_h} = f_h + O(h) \ \Rightarrow \ \|\xi_h - z_{\Delta_h}\| \le S \, h$$

Difficulties

Consider $\quad \Phi_h \xi_h = f_h, \;\; \Phi_h z_{\Delta_h} = f_h + O(h^2)$

Then $\quad \|\xi_h - z_{\Delta_h}\| \leq S\,\|\Phi_h\xi_h - \Phi_h z_{\Delta_h}\| \leq S\,h^2$

- Difficulty in singular case

$$(\Phi_h z_{\Delta_h})_i = \frac{z(t_i) - z(t_{i-1})}{h} - \frac{D}{t_{i-1/2}} \frac{z(t_i) + z(t_{i-1})}{2} = f_{i-1/2} + O\left(\frac{h^2}{t_{i-1/2}}\right)$$

$$\Phi_h \xi_h = f_h, \;\; \Phi_h z_{\Delta_h} = f_h + O(h) \;\Rightarrow\; \|\xi_h - z_{\Delta_h}\| \leq S\,h$$

- Remedy: $(\Phi_h(\xi_h - z_{\Delta_h}))_i = O\left(\frac{h^2}{t_{i-1/2}}\right) \;\Rightarrow$

$$(\xi_h - z_{\Delta_h})_i = \Phi_h^{-1} O\left(\frac{h^2}{t_{i-1/2}}\right) = O(h^2)$$

Problem setting: IVP

Consider the following linear system and assume that $\lambda(M) \in \mathbb{R}$ and $\lambda < 0$,

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0, 1],$$
$$z(0) = 0.$$

Problem setting: IVP

Consider the following linear system and assume that $\lambda(M) \in \mathbb{R}$ and $\lambda < 0$,

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0,1],$$
$$z(0) = 0.$$

We decouple the problem by using $M = EJE^{-1}$.

Problem setting: IVP

Consider the following linear system and assume that $\lambda(M) \in \mathbb{R}$ and $\lambda < 0$,

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0,1],$$
$$z(0) = 0.$$

We decouple the problem by using $M = EJE^{-1}$.

With $v(t) := E^{-1}z(t),\ \ g(t) := E^{-1}f(t)$, the problem becomes

$$z'(t) = \frac{1}{t} EJE^{-1} z(t) + f(t) \quad \Leftrightarrow$$

Problem setting: IVP

Consider the following linear system and assume that $\lambda(M) \in \mathbb{R}$ and $\lambda < 0$,

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0,1],$$
$$z(0) = 0.$$

We decouple the problem by using $M = EJE^{-1}$.

With $v(t) := E^{-1}z(t),\ \ g(t) := E^{-1}f(t)$, the problem becomes

$$z'(t) = \frac{1}{t} EJE^{-1} z(t) + f(t) \quad \Leftrightarrow$$

$$\underbrace{E^{-1}z'(t)}_{=:v'(t)} = \frac{1}{t} J \underbrace{E^{-1}z(t)}_{=:v(t)} + \underbrace{E^{-1}f(t)}_{=:g(t)}, \quad v(0) = E^{-1}z(0) = 0.$$

Decoupled problem

Now the problem reads

$$v'(t) = \frac{1}{t} J v(t) + g(t), \quad t \in (0,1], \qquad v(0) = 0.$$

Decoupled problem

Now the problem reads

$$v'(t) = \frac{1}{t} J v(t) + g(t), \quad t \in (0,1], \qquad v(0) = 0.$$

Consider the operator F defined as

$$F(v) := \begin{pmatrix} v'(t) - \frac{1}{t} J v(t) - g(t), \quad t \in (0,1] \\ v(0) \end{pmatrix},$$

then for the exact solution $F(v)$ = 0 holds.

Numerical scheme: Forward Euler

Consider a uniform mesh,

$$\Delta_h := (t_{j_0}, t_{j_0+1}, \ldots, t_N), \quad t_i = ih, \; i = j_0, \ldots, N, \; h = \frac{1}{N}, \; j_0 > 0$$

and $\lim_{h \to 0} t_{j_0} = 0$.

Numerical scheme: Forward Euler

Consider a uniform mesh,

$$\Delta_h := (t_{j_0}, t_{j_0+1}, \dots, t_N), \quad t_i = ih, \ \ i = j_0, \dots, N, \ \ h = \frac{1}{N}, \ \ j_0 > 0$$

and $\lim_{h \to 0} t_{j_0} = 0$.

Applying the forward Euler method to the problem

$$v'(t) = \frac{1}{t} J v(t) + g(t), \quad t \in (0,1], \qquad v(0) = 0,$$

we obtain

$$\begin{aligned} v_{j_0} &:= v(0), \\ v_{j+1} &:= v_j + h\Big(\frac{1}{t_j} J v_j + g(t_j)\Big), \quad j = j_0, \dots, N-1. \end{aligned}$$

Numerical scheme

Let $v_h = (v_{j_0}, \ldots, v_N)$ and consider the discrete operator

$$F_h(v_h) := \begin{pmatrix} \dfrac{v_{j+1} - v_j}{h} - \dfrac{1}{t_j} J v_j - g(t_j) \quad j = j_0, \ldots, N-1 \\ v_{j_0} - v(0) \end{pmatrix},$$

Numerical scheme

Let $v_h = (v_{j0}, \ldots, v_N)$ and consider the discrete operator

$$F_h(v_h) := \begin{pmatrix} \dfrac{v_{j+1} - v_j}{h} - \dfrac{1}{t_j} J v_j - g(t_j) \quad j = j_0, \ldots, N-1 \\ v_{j_0} - v(0) \end{pmatrix},$$

then the numerical solution satisfies $F_h(v_h) = 0$.

Note: We set $v_{j_0} = v(0)$ not $v_{j_0} = v(t_{j_0})$.

Numerical scheme

Let $v_h = (v_{j_0}, \ldots, v_N)$ and consider the discrete operator

$$F_h(v_h) := \begin{pmatrix} \dfrac{v_{j+1} - v_j}{h} - \dfrac{1}{t_j} J v_j - g(t_j) \quad j = j_0, \ldots, N-1 \\ v_{j_0} - v(0) \end{pmatrix},$$

then the numerical solution satisfies $F_h(v_h) = 0$.

Note: We set $v_{j_0} = v(0)$ not $v_{j_0} = v(t_{j_0})$.

Recall

$$F(v) := \begin{pmatrix} v'(t) - \dfrac{1}{t} J v(t) - g(t), \quad t \in (0,1] \\ v(0) \end{pmatrix}, \qquad F(v) = 0.$$

Numerical scheme

Let $v_h = (v_{j_0}, \dots, v_N)$ and consider the discrete operator

$$F_h(v_h) := \begin{pmatrix} \dfrac{v_{j+1} - v_j}{h} - \dfrac{1}{t_j} J v_j - g(t_j) \quad j = j_0, \dots, N-1 \\ v_{j_0} - v(0) \end{pmatrix},$$

then the numerical solution satisfies $F_h(v_h) = 0$.

Note: We set $v_{j_0} = v(0)$ not $v_{j_0} = v(t_{j_0})$.

Recall

$$F(v) := \begin{pmatrix} v'(t) - \dfrac{1}{t} J v(t) - g(t), \quad t \in (0,1] \\ v(0) \end{pmatrix}, \qquad F(v) = 0.$$

Aim: Show the convergence of the approximate solution v_h towards the exact solution v as $h \to 0$.

Towards convergence

Prove that the family F_h is stable and consistent by decoupling the system into single equations.

Towards convergence

Prove that the family F_h is stable and consistent by decoupling the system into single equations.

Recall that J is composed of Jordan blocks

$$J = \begin{pmatrix} \lambda_1 & 1 & & & & & \\ & \lambda_1 & 1 & & & & \\ & & \ddots & 1 & & & \\ & & & \lambda_1 & & & \\ & & & & \lambda_2 & 1 & \\ & & & & & \ddots & \end{pmatrix}.$$

Towards convergence

Consider the system $v'(t) = \frac{J}{t} v(t) + g(t)$, where

$$J = \begin{pmatrix} \lambda & 1 & \\ & \ddots & 1 \\ & & \lambda \end{pmatrix}, \quad J \in \mathbb{R}^{k \times k}, \; \lambda < 0.$$

Towards convergence

Consider the system $v'(t) = \frac{J}{t} v(t) + g(t)$, where

$$J = \begin{pmatrix} \lambda & 1 & & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}, \quad J \in \mathbb{R}^{k \times k}, \ \ \lambda < 0.$$

Then the system consists of two types of equations (T2.6),

Towards convergence

Consider the system $v'(t) = \frac{J}{t} v(t) + g(t)$, where

$$J = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & 1 & \\ & & \lambda & \end{pmatrix}, \quad J \in \mathbb{R}^{k \times k}, \quad \lambda < 0.$$

Then the system consists of two types of equations (T2.6),

$$v_k'(t) = \frac{\lambda}{t} v_k(t) + g_k(t),$$

Towards convergence

Consider the system $v'(t) = \frac{J}{t} v(t) + g(t)$, where

$$J = \begin{pmatrix} \lambda & 1 & & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}, \quad J \in \mathbb{R}^{k \times k}, \;\; \lambda < 0.$$

Then the system consists of two types of equations (T2.6),

$$v_k'(t) = \frac{\lambda}{t} v_k(t) + g_k(t),$$

$$v_i'(t) = \frac{\lambda}{t} v_i(t) + \frac{1}{t} v_{i+1}(t) + g_i(t), \quad i = 1, \dots, k-1.$$

Stability

We first consider the equation

$$v'(t) - \frac{\lambda}{t} v(t) - g(t) = 0, \quad v(0) = 0,$$

and analyze its discretization

$$F_h(v_h) = \left(\frac{v_{j+1} - v_j}{h} - \frac{\lambda}{t_j} v_j - g(t_j),\ j = j_0, \ldots, N-1,\ v_{j_0} - v(0) \right) = 0,$$

Stability

We first consider the equation

$$v'(t) - \frac{\lambda}{t} v(t) - g(t) = 0, \quad v(0) = 0,$$

and analyze its discretization

$$F_h(v_h) = \left(\frac{v_{j+1} - v_j}{h} - \frac{\lambda}{t_j} v_j - g(t_j),\ j = j_0, \ldots, N-1,\ v_{j_0} - v(0) \right) = 0,$$

Show the stability: Consider two systems,

$$F_h(u_h) = \delta_h^1, \quad F_h(w_h) = \delta_h^2 \Rightarrow F_h(u_h - w_h) = \delta_h^1 - \delta_h^2,$$

and show $\|u_h - w_h\| \leq K \|\delta_h^1 - \delta_h^2\| = K \|F_h(u_h) - F_h(w_h)\|$.

Stability

We first consider the equation

$$v'(t) - \frac{\lambda}{t} v(t) - g(t) = 0, \quad v(0) = 0,$$

and analyze its discretization

$$F_h(v_h) = \left(\frac{v_{j+1} - v_j}{h} - \frac{\lambda}{t_j} v_j - g(t_j),\ j = j_0, \ldots, N-1,\ v_{j_0} - v(0) \right) = 0,$$

Show the stability: Consider two systems,

$$F_h(u_h) = \delta_h^1, \quad F_h(w_h) = \delta_h^2 \Rightarrow F_h(u_h - w_h) = \delta_h^1 - \delta_h^2,$$

and show $\|u_h - w_h\| \le K\|\delta_h^1 - \delta_h^2\| = K\|F_h(u_h) - F_h(w_h)\|$.

Introduce $\varepsilon_h := u_h - w_h, \quad l_h := \delta_h^1 - \delta_h^2$.

Stability

Consequently ε_h satisfies

$$\frac{\varepsilon_{j+1}-\varepsilon_j}{h}-\frac{\lambda}{t_j}\varepsilon_j=l_{j+1}\quad j=j_0,\dots,N-1,\quad \varepsilon_{j_0}=l_{j_0}.$$

Stability

Consequently ε_h satisfies

$$\frac{\varepsilon_{j+1} - \varepsilon_j}{h} - \frac{\lambda}{t_j}\varepsilon_j = l_{j+1} \quad j = j_0, \ldots, N-1, \quad \varepsilon_{j_0} = l_{j_0}.$$

The aim is now to show that

$\|\varepsilon_h\|$ is bounded in terms of $\|l_h\| = \max_{j_0 \le i \le N} |l_h|$,

$$\|\varepsilon_h\| \le \text{const}\|l_h\|.$$

Stability

Lemma: The solution of the scheme (T2.7-T2.9)

$$\frac{\varepsilon_{j+1}-\varepsilon_j}{h}-\frac{\lambda}{t_j}\varepsilon_j = l_{j+1} \quad j = j_0,\dots,N-1, \quad \varepsilon_{j_0} = l_{j_0}.$$

is given by

$$\varepsilon_j = \prod_{l=j_0}^{j-1}\left(1+\frac{h\lambda}{t_l}\right)l_{j_0} + \sum_{l=j_0}^{j-2}\prod_{k=l+1}^{j-1}\left(1+\frac{h\lambda}{t_k}\right)hl_{l+1} + hl_j.$$

Stability

Lemma: The solution of the scheme (T2.7-T2.9)

$$\frac{\varepsilon_{j+1}-\varepsilon_j}{h} - \frac{\lambda}{t_j}\varepsilon_j = l_{j+1} \quad j = j_0, \dots, N-1, \quad \varepsilon_{j_0} = l_{j_0}.$$

is given by

$$\varepsilon_j = \prod_{l=j_0}^{j-1} \left(1 + \frac{h\lambda}{t_l}\right) l_{j_0} + \sum_{l=j_0}^{j-2} \prod_{k=l+1}^{j-1} \left(1 + \frac{h\lambda}{t_k}\right) h l_{l+1} + h l_j.$$

Moreover, the following estimates hold:

$$|\varepsilon_j| \leq \tilde{K}(|l_{j_0}| + t_j \max_{j_0+1 \leq l \leq N} |l_l|) \leq K\|l_h\|, \quad j = j_0, \dots, N.$$

Proof.

Stability

Auxiliary Lemma 1: Let $\lambda < 0$ and let for $j \geq k \geq 1$

$$z_{kj}(\lambda) := \prod_{l=k}^{j-1} \left(1 + \frac{h\lambda}{t_l}\right), \quad 1 \leq k < j, \quad j = 2, 3, \ldots, \qquad z_{jj} := 1.$$

Then, there exist constants $\eta > 0$ and $C \geq 1$ such that

$$|z_{kj}(\lambda)| \leq C \left(\frac{t_k}{t_j}\right)^\eta = C \left(\frac{k}{j}\right)^\eta, \quad 1 \leq k \leq j, \quad j = 1, 2, 3, \ldots.$$

Proof. (T2.10-T2.11)

Stability

Auxiliary Lemma 2: Let $h > 0$, $t_j := jh$, $k > j \geq i_0$ and $\gamma \in \mathbb{R}$. Then,

$$\sum_{l=j}^{k-1} h t_l^{\gamma-1} \leq \begin{cases} \text{const}\left|t_k^\gamma - t_j^\gamma\right|, & \gamma \neq 0, \\ \\ \text{const}\ln\left(\frac{t_k}{t_j}\right), & \gamma = 0. \end{cases}$$

Proof. (T2.12)

Stability estimate

Lemma: The solution of the scheme

$$\frac{\varepsilon_{j+1}-\varepsilon_j}{h} - \frac{\lambda}{t_j}\varepsilon_j = l_{j+1} \quad j = j_0, \ldots, N-1, \quad \varepsilon_{j_0} = l_{j_0}.$$

is given by

$$\varepsilon_j = \prod_{l=j_0}^{j-1}\left(1+\frac{h\lambda}{t_l}\right) l_{j_0} + \sum_{l=j_0}^{j-2}\prod_{k=l+1}^{j-1}\left(1+\frac{h\lambda}{t_k}\right) h l_{l+1} + h l_j.$$

Stability estimate

Lemma: The solution of the scheme

$$\frac{\varepsilon_{j+1}-\varepsilon_j}{h} - \frac{\lambda}{t_j}\varepsilon_j = l_{j+1} \quad j = j_0, \dots, N-1, \quad \varepsilon_{j_0} = l_{j_0}.$$

is given by

$$\varepsilon_j = \prod_{l=j_0}^{j-1}\left(1+\frac{h\lambda}{t_l}\right) l_{j_0} + \sum_{l=j_0}^{j-2}\prod_{k=l+1}^{j-1}\left(1+\frac{h\lambda}{t_k}\right) h l_{l+1} + h l_j.$$

Moreover, the following estimates hold:

$$|\varepsilon_j| \le \tilde{K}\left(|l_{j_0}| + t_j \max_{j_0+1\le l\le N} |l_l|\right) \le K\|l_h\|, \quad j = j_0, \dots, N.$$

Proof. (T2.13-T2.14)

Consistency

Theorem: Let $v \in C^2[0,1]$, be the solution of $F(v) = 0$, then $v(0) = 0$. Define

$$l_h := F_h(v_{\Delta_h}).$$

Consistency

Theorem: Let $v \in C^2[0,1]$, be the solution of $F(v) = 0$, then $v(0) = 0$. Define

$$l_h := F_h(v_{\Delta_h}).$$

Then

$$l_h = \begin{pmatrix} \dfrac{v(t_{j+1}) - v(t_j)}{h} - v'(t_j), \quad j = j_0, \ldots, N-1 \\ v(t_{j_0}) - v(0) \end{pmatrix}.$$

Consistency

Theorem: Let $v \in C^2[0,1]$, be the solution of $F(v) = 0$, then $v(0) = 0$. Define

$$l_h := F_h(v_{\Delta_h}).$$

Then

$$l_h = \begin{pmatrix} \dfrac{v(t_{j+1}) - v(t_j)}{h} - v'(t_j), \quad j = j_0, \ldots, N-1 \\ v(t_{j_0}) - v(0) \end{pmatrix}.$$

For $v \in C^2[0,1]$, we have

$$\|l_h\| = O(h).$$

Proof. (T2.15)

Convergence

Corollary
The sequence of solutions v_h of the problems $F_h(v_h) = 0$ converges towards v_{Δ_h}, where v is the exact solution of $F(v) = 0$.
For $v \in C^2[0,1]$ the convergence order is one.

Examples

Example 1: Consider the linear problem

$$y''(t) = -\frac{2}{t}y'(t) \underbrace{-n^2 \cos(nt) - \frac{2}{t} n \sin(nt)}_{\text{has a limit for } t \to 0}, \quad t \in (0, 1],$$

$$y(0) = 2, \quad y'(0) = 0,$$

where $y(t) = 1 + \cos(nt), \; n = 3.$

Examples

Example 1: Consider the linear problem

$$y''(t) = -\frac{2}{t} y'(t) \underbrace{-n^2 \cos(nt) - \frac{2}{t} n \sin(nt),}_{\text{has a limit for } t \to 0} \quad t \in (0, 1],$$

$$y(0) = 2, \quad y'(0) = 0,$$

where $y(t) = 1 + \cos(nt), \;\; n = 3.$

Example 2: Consider the nonlinear problem

$$y''(t) = -\frac{2}{t} y'(t) - y^5(t), \quad t \in (0, 1],$$
$$y(0) = 1, \quad y'(0) = 0,$$

where $y(t) = \frac{1}{\sqrt{1+t^2/3}}$.

Example 1

h	$\|\varepsilon_h\|$	p	c
$1/5$	8.5	0.970	$-4.0 \cdot 10^{+01}$
$1/5 \cdot 2^{-1}$	4.3	0.982	$-4.1 \cdot 10^{+01}$
$1/5 \cdot 2^{-2}$	$2.1 \cdot 10^{-01}$	0.990	$-4.2 \cdot 10^{+01}$
$1/5 \cdot 2^{-3}$	$1.1 \cdot 10^{-01}$	0.995	$-4.3 \cdot 10^{+01}$
$1/5 \cdot 2^{-4}$	$5.5 \cdot 10^{-01}$	0.997	$-4.3 \cdot 10^{+01}$
$1/5 \cdot 2^{-5}$	$2.7 \cdot 10^{-02}$	0.998	$-4.4 \cdot 10^{+01}$
$1/5 \cdot 2^{-6}$	$1.3 \cdot 10^{-02}$	0.999	$-4.4 \cdot 10^{+01}$
$1/5 \cdot 2^{-7}$	$6.9 \cdot 10^{-02}$	0.999	$-4.4 \cdot 10^{+01}$
$1/5 \cdot 2^{-8}$	$3.4 \cdot 10^{-02}$	0.999	$-4.4 \cdot 10^{+01}$
$1/5 \cdot 2^{-9}$	$1.7 \cdot 10^{-03}$	0.999	$-4.4 \cdot 10^{+01}$

Example 2

h	$\|\varepsilon_h\|$	p	c
$1/5$	$2.4 \cdot 10^{-02}$	0.834	$-9.2 \cdot 10^{-01}$
$1/5 \cdot 2^{-1}$	$1.3 \cdot 10^{-02}$	0.921	$-1.1 \cdot 10^{-01}$
$1/5 \cdot 2^{-2}$	$7.1 \cdot 10^{-02}$	0.960	$-1.2 \cdot 10^{-01}$
$1/5 \cdot 2^{-3}$	$3.6 \cdot 10^{-02}$	0.980	$-1.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-4}$	$1.8 \cdot 10^{-03}$	0.990	$-1.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-5}$	$9.3 \cdot 10^{-03}$	0.995	$-1.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-6}$	$4.7 \cdot 10^{-03}$	0.997	$-1.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-7}$	$2.3 \cdot 10^{-04}$	0.998	$-1.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-8}$	$1.1 \cdot 10^{-04}$	0.999	$-1.5 \cdot 10^{-01}$
$1/5 \cdot 2^{-9}$	$5.8 \cdot 10^{-04}$	0.999	$-1.5 \cdot 10^{-01}$

Summary

We discussed convergence of the forward Euler method applied to

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0, 1], \quad z(0) = 0, \quad \lambda(M) < 0.$$

Summary

We discussed convergence of the forward Euler method applied to

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0,1], \quad z(0) = 0, \quad \lambda(M) < 0.$$

- For $z \in C^2[0,1]$, forward Euler method shows classical convergence order one.

Summary

We discussed convergence of the forward Euler method applied to

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0,1], \quad z(0) = 0, \quad \lambda(M) < 0.$$

- For $z \in C^2[0,1]$, forward Euler method shows classical convergence order one.
- For stability we invert the discrete operator.

Summary

We discussed convergence of the forward Euler method applied to

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0, 1], \quad z(0) = 0, \quad \lambda(M) < 0.$$

- For $z \in C^2[0, 1]$, forward Euler method shows classical convergence order one.
- For stability we invert the discrete operator.
- Consistency of order $p = 1$ holds. Therefore that classical approach is applicable and the global error converges of order one on the interval $[t_0, 1]$, where $t_0 \to 0$ for $h \to 0$.