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Numerical Treatment of Singular BVPs in ODEs

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Basic Concepts: Stability of the Nonlinear Operator

H.B. Keller: Approximation Methods for Nonlinear Problems with application to Two-Point BVPs, *Math. Comp.* **29**, pp. 464-474 (1975)

Nonlinear operator equation,

$$F(x) = 0, \quad F : B_1 \to B_2,$$

were $B_1, \ B_2$ are Banach spaces.

Aim: Study the properties of the related discrete problem

$$F_h(x_h) = 0, \quad F_h : B_{1,h} \to B_{2,h},$$

where h is a discretization parameter (step size).

Main result - analytical problem

Let $F(x) = 0, \;\; F : B_1 \to B_2$.

Consider the sphere $S_\rho(u) := \{x : x \in B_1, \|x - u\| \le \rho\}.$

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Definition: *The mapping $F(\cdot)$ is stable on $S_\rho(u)$ iff there exists a constant $K_\rho > 0$ such that*

$$\|v - w\| \leq K_\rho \|F(v) - F(w)\|,$$

for all $v, w \in S_\rho(u)$.

A solution $x = u$ of $F(x) = 0$ is stable iff $F(\cdot)$ is stable on $S_\rho(u)$ for some $\rho > 0$.

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A solution $x = u$ of $F(x) = 0$ is stable iff $F(\cdot)$ is stable on $S_\rho(u)$ for some $\rho > 0$.

- Stable solution u is unique in $S_\rho(u)$.
- Let F be linear, and $Fv = g$, then

  v depends continuously on g, $\|v\| \leq K_\rho \|g\|$.

Main result - analytical problem

Let $F : B_1 \to B_2$ and let $x = u$ be a solution of $F(x) = 0$.
Consider the linearization of $F(x) = 0$ at u:

$$L(u)y = 0, \;\; L(u) : B_1 \to B_2,$$

where $L(x)$ is the Fréchet derivative of F at point x.
The operator $L(x)$ is a linear operator, $L(x) : B_1 \to B_2$ such that

$$r(x, y) = \frac{\|F(x+y) - [F(x) + L(x)y]\|}{\|y\|} \to 0 \text{ as } \|y\| \to 0.$$

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The above conditions mean that $L(u)$ is injective. From now on, we assume that $L(u) : \mathcal{D}(L(u)) = B_1 \to \mathcal{R}(L(u)) = B_2$ and so $L(u)$ is bijective and $L^{-1}(u)$ exists and is bounded.

Analytical problem: stable vs. isolated solution

Theorem 1. Let u be a stable solution of $F(x) = 0$. Then, if $L(u)$ exists, it is nonsingular and consequently u is isolated.

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Theorem 2. Let $u \in B_1$ be isolated, which means that $L(u)$ is nonsingular. Let $L(u)$ have a bounded inverse.

Let $L(x)$ exist and be Lipschitz continous on $S_{\rho_0}(u)$ for a $\rho_0 > 0$:

This means that for some constant $K_L > 0$,

$$\|L(x) - L(y)\| \leq K_L \|x - y\|$$

for all $x, y \in S_{\rho_0}(u)$.

Then, $F(\cdot)$ is stable on $S_\rho(u)$ for a sufficiently small ρ.

Focus: Isolated solution

Let $B_1 := C^1[0,1]$ and $B_2 = C[0,1] \times \mathbb{R}^n$

$$F(y)(t) := \left\{ y'(t) - \left(\frac{M}{t} y(t) + f(t, y(t)) \right) = 0,\ b(y(0), y(1)) = 0. \right.$$

have an isolated solution. This means that the linear problem,

$$L(y)v(t) := \left\{ v'(t) - \frac{A(t)}{t} v(t) = 0,\ B_0 v(0) + B_1 v(1) = 0, \right.$$

where $L(y) : B_1 := C^1[0,1] \to B_2 = C[0,1] \times \mathbb{R}^n$, and

$$A(t) = M + t \frac{\partial f(t, y(t))}{\partial y}, \quad B_0 = \frac{\partial b(y(0), y(1))}{\partial y(0)}, \quad B_1 = \frac{\partial b(y(0), y(1))}{\partial y(1)}$$

has only the trivial solution.

Focus: Unique solution

We derive conditions for the the unique solvability of the singular problem

$$L(y)v(t) := \left\{ v'(t) - \frac{A(t)}{t} v(t) = f(t),\; B_0 v(0) + B_1 v(1) = \beta, \right.$$

where

$$L(y) : B_1 := C^1[0,1] \to B_2 = C[0,1] \times \mathbb{R}^n$$

and

$$A(t) \in C^1[0,1], \;\; B_0, B_1 \in \mathbb{R}^{n\times n}, \; \beta \in \mathbb{R}^n.$$

Discretizations of the Nonlinear Operator Equations

H.B. Keller (1975)

On the family of Banach spaces $\{B_1^h, B_2^h\}$, we consider the family of approximating problems, $0 < h \leq h_0$:

$$F_h(x_h) = 0, \quad F_h : B_1^h \to B_2^h.$$

Example: We consider the following nonlinear scalar problem:

$$N(y) = \begin{cases} y'(t) - f(t, y(t)), & t \in [0,1], \\ a_1 y(0) + a_2 y(1) - a_3, \end{cases}$$

where $y : [0,1] \to \mathbb{R}$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$.

Discretization of the Nonlinear Operator Equations

A simple example of a family of approximating problems: forward Euler scheme.
Introduce a mesh on the interval $[0, 1]$,

$$\Delta_h := \{t_i \,|\, t_i = ih,\ i = 0, 1, \dots, N,\ h = t_{i+1} - t_i\}.$$

Then the scheme is

$$N_h(y_h) = \begin{cases} \frac{y_{i+1}-y_i}{h} - f\left(t_i, y_i\right) = 0, & i = 0, \dots, N-1, \\ a_1 y_0 + a_2 y_N - a_3 = 0, \end{cases}$$

where $y_i \approx y(t_i)$ and $y_h = (y_0, y_1, \dots, y_N)^T$.

Discretization of the Nonlinear Operator Equations

To relate the problems $F(x) = 0$ and $F_h(x_h) = 0$, we require the existence of a family of linear mappings $\{P_1^h, P_2^h\}$, such that

$$\text{(a)} \ \ P_\nu^h : B_\nu \to B_\nu^h, \qquad \text{(b)} \ \ \lim_{h \to 0} \|P_\nu^h x\| = \|x\|, \ \ \forall x \in B_\nu.$$

We use the notation

$$P_\nu^h x := [x]_h, \ \nu = 1, 2,$$

where, $[x]_h \in B_\nu^h$ if $x \in B_\nu$. In our example,

$$[y]_h = (y(t_0), \ldots, y(t_N))^T.$$

Discretization of the Nonlinear Operator Equations

Let $x = u$ be the solution of the analytical problem $F(x) = 0$
and
let $x_h = v_h$ be the solution of the scheme $F_h(x_h) = 0$.

The aim is to study the so-called global discretization error,

$$\|[u]_h - v_h\|$$

with appropriately chosen norm in B_1^h.

The Fréchet derivative of F_h at x_h is denoted by $L_h(x_h)$, and $S_\rho(x_h)$ is the sphere in B_1^h of radius ρ about x_h.

We always assume that a stable (and therefore locally unique)

solution $u \in B_1$ with $F(u) = 0$ exists.

Discretization: Stability and Consistency

$$F(x) = 0, \;\; S_\rho(u), \qquad F_h(x_h) = 0, \;\; S_\rho([u]_h).$$

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Definition: The family $\{F_h(\cdot)\}$ is stable for $u \in B_1$, iff for some $h_0 > 0$, $\rho > 0$ and some constant M_ρ, independent of h,

$$\|x_h - y_h\| \leq M_\rho \|F_h(x_h) - F_h(y_h)\|,$$

for all $x_h, y_h \in S_\rho([u]_h)$ and all $0 < h \leq h_0$.

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for all $x_h, y_h \in S_\rho([u]_h)$ and all $0 < h \le h_0$.

Definition: The family $\{F_h(\cdot)\}$ is consistent of order p with $F(\cdot)$ on $S_\rho(u)$, iff

$$\|F_h([x]_h) - [F(x)]_h\| := \|\tau_h(x)\| \le M(x)h^p$$

for all $x \in S_\rho(u)$ and some bounded functional $M(x) \ge 0$ independent of h.

Discretization: Stability and Consistency

We now interpret the last definition in context of our example:

$$N_h([y]_h) - [N(y)]_h = ?$$

$$N_h([y]_h) = \begin{cases} \frac{y(t_{i+1}) - y(t_i)}{h} - f(t_i, y(t_i)), & 0 \le i \le N-1, \\ a_1 y(0) + a_1 y(1) - a_3, & \end{cases}$$

$$[N(y)]_h = \begin{cases} y'(t_i) - f(t_i, y(t_i)), & 0 \le i \le N-1, \\ a_1 y(0) + a_1 y(1) - a_3, & \end{cases}$$

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which means

$$N_h([y]_h) - [N(y)]_h = \begin{cases} \frac{y(t_{i+1})-y(t_i)}{h} - y'(t_i), & 0 \le i \le N-1, \\ 0. & \end{cases}$$

Discretization: Stability and Consistency

Note that for the solution y^*, with $N(y^*) = 0$,

$$N_h([y^*]_h) - [N(y^*)]_h = \begin{cases} \frac{y^*(t_{i+1}) - y^*(t_i)}{h} - (y^*)'(t_i), & 0 \le i \le N-1, \\ 0. \end{cases}$$

Discretization: Stability and Consistency

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The quantity $\tau_h(y^*) = N_h([y^*]_h) - [N(y^*)]_h = N_h([y^*]_h)$ is called *residual*, sometimes also *local discretization error*.

Discretization: Stability and Consistency

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The quantity $\tau_h(y^*) = N_h([y^*]_h) - [N(y^*)]_h = N_h([y^*]_h)$ is called *residual*, sometimes also *local discretization error*.

We now use the stability to 'sum up' the

local discretization errors $\tau_h(y^*)$

in order to obtain an à priori bound for the

global discretization error $\|[y^*]_h - y_h\|$.

Stability + Consistency = Convergence

Theorem 3. Let $F(u) = 0$ and $F_h(v_h) = 0$ for some

$$v_h \in S_\rho([u]_h), \quad \rho > 0 \;\text{ and all }\; 0 < h \leq h_0.$$

Let $\{F_h(\cdot)\}$ be stable for u and consistent with $F(\cdot)$ of order p on $S_0(u)$. Then

$$\|[u]_h - v_h\| \leq M_\rho M(u) h^p.$$

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$$\|[u]_h - v_h\| \leq M_\rho M(u) h^p.$$

Proof. We consider two schemes

$$F_h(x_h) := F_h([u]_h) = \tau_h(u), \quad F_h(y_h) := F_h(v_h) = 0$$

and use the *stability and consistency* to obtain,

$$\|[u]_h - v_h\| \leq M_\rho \|F_h([u]_h) - F_h(v_h)\| = M_\rho \|\tau_h(u)\| \leq M_\rho M(u) h^p.$$

Stability + Consistency = Convergence: Questions

- Does the approximating problem $F_h(v_h) = 0$ have a solution in some sphere $S_\rho([u]_h)$,

$$v_h \in S_\rho([u]_h), \quad \rho > 0 \ \text{and all} \ 0 < h \le h_0?$$

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- Can we verify stability?

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- Can we determine the order of consistency?

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- Can we verify stability?
- Can we determine the order of consistency?

The answer depends on the schemes we are considering. Stability of explicit difference schemes can be often easily verified, but it is hard to verify stability for implicit schemes. Quite often consistency can be investigated by a simple Taylor argument. In case of forward Euler and $y^* \in C^2$, it immediately follows

$$\tau_h(y^*) = N_h([y^*]_h) - [N(y^*)]_h = \begin{cases} \frac{y^*(t_{i+1}) - y^*(t_i)}{h} - (y^*)'(t_i) = \frac{h}{2}(y^*)''(\eta_i), & 0 \leq i \leq N-1, \\ 0, & \end{cases}$$

with order of consistency $p = 1$. In general, the stability analysis is restricted to the study of the linearized problems.

Main results: Stability of the discretization

Sufficient conditions for $\{F_h(\cdot)\}$ to be stable.

Theorem 4. Let the family of mappings $\{F_h(\cdot)\}$ have Fréchet derivatives, i.e., linearizations, $\{L_h(x_h)\}$ on some family of spheres $S_{\rho_0}(z_h)$ and satisfy for all $0 < h \in h_0$:

(a) $\{L_h(z_h)\}$ have uniformly bounded inverses at the centers of the spheres; that is, for some constant $K_0 > 0$, $\|L_h^{-1}(z_h)\| \leq K_0$.

(b) $\{L_h(x_h)\}$ are uniformly Lipschitz continuous on $S_{\rho_0}(z_h)$; that is, for some constant $K_L > 0$,

$$\|L_h(x_h) - L_h(y_h)\| \leq K_L \|x_h - y_h\| \ \text{for all} \ x_h, y_h \in S_{\rho_0}(z_h).$$

If $z_h = [u]_h$ for some $u \in B_1$, then family $\{F_h(\cdot)\}$ is stable for u.

Proof. (T1.1 - T1.4)

Main results: Solutions to $\{F_h(v_h) = 0\}$

Sufficient conditions for the solution to $\{F_h(v_h) = 0\}$ to exist.

Theorem 5. Let $x = u$ be a solution of $F(x) = 0$.

Let (a) and (b) from Theorem 4 hold with $z_h = [u]_h$.

Let family $\{F_h(\cdot)\}$ be consistent of order p with $F(\cdot)$ on $S_0(u)$.

Then, for sufficiently small ρ_0 and h_0, and each $0 < h \leq h_0$, the problem $F_h(x_h) = 0$ has a unique solution $x_h = v_h \in S_{\rho_0}([u]_h)$.

This solution v_h satisfies

$$\|[u]_h - v_h\| \leq M_{\rho_0} M(u) h^p.$$

Proof. (T1.5 - T1.9)

Main results: Newton iteration

Theorem 6. Let the hypothesis of Theorem 5 hold.

Then, for any $0 < h \leq h_0$, if $\rho_0,\ h_0$ and $\rho_1 \leq \rho_0$ are sufficiently small, the Newton iterates $\{v_h^{(\nu)}\}$ defined by

(a) $v_h^{(0)} \in S_{\rho_1}([u]_h)$,

(b) $L_h(v_h^{(\nu)})[v_h^{(\nu+1)} - v_h^{(\nu)}] = -F_h(v_h^{(\nu)}),\ \nu = 0, 1, \ldots,$

converge quadratically to the

solution v_h of $F_h(x_h) = 0$, $v_h \in S_{\rho_0}([u]_h)$.

Proof. (T1.10 - T1.16)

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- Family of approximations $\{F_h(\cdot)\}$ is stable for u and consistent of order p with $F(\cdot) = 0$ on $S_{\rho_0}([u]_h)$.

Summary

- Nonlinear problem $F(x) = 0$ has an isolated solution u.
- Family of approximations $\{F_h(\cdot)\}$ is stable for u and consistent of order p with $F(\cdot) = 0$ on $S_{\rho_0}([u]_h)$.
- The there exists a step-size h_0 such that for all $0 < h \leq h_0$ the approximating problem $F_h(x_h) = 0$ has a solution v_h in $S_{\rho_0}([u]_h)$, provided ρ_0 is sufficiently small.

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- The there exists a step-size h_0 such that for all $0 < h \leq h_0$ the approximating problem $F_h(x_h) = 0$ has a solution v_h in $S_{\rho_0}([u]_h)$, provided ρ_0 is sufficiently small.
- For $h \to 0$ this solution v_h converges to $[u]_h$ and the global error is of order p, $\|v_h - [u]_h\| = O(h^p)$.

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- The there exists a step-size h_0 such that for all $0 < h \leq h_0$ the approximating problem $F_h(x_h) = 0$ has a solution v_h in $S_{\rho_0}([u]_h)$, provided ρ_0 is sufficiently small.
- For $h \to 0$ this solution v_h converges to $[u]_h$ and the global error is of order p, $\|v_h - [u]_h\| = O(h^p)$.
- Moreover, there exists a constant $\rho_1 \leq \rho_0$ such that the Newton iteration converges quadratically in $S_{\rho_0}([u]_h)$ from any starting value $v_h^{(0)} \in S_{\rho_1}([u]_h)$, provided ρ_1 is sufficiently small.