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A Domain Decomposition Algorithm for Contact Problems with Coulomb's Friction

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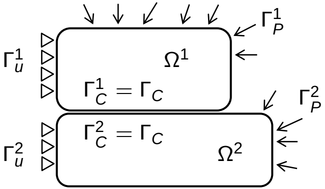
Outline

Outline

PDEs



- two bodies Ω^1, Ω^2 in $\mathbb{R}^2$

$$\partial\Omega^\alpha = \overline{\Gamma}_u^\alpha \cup \overline{\Gamma}_P^\alpha \cup \overline{\Gamma}_C^\alpha$$

- zero displacement on Γ_u^α
- surface traction on Γ_P^α
- contact conditions on Γ_C

- the Lamè PDEs in Ω^α, $\alpha = 1, 2$:

$$\begin{aligned} -\operatorname{div} \sigma^\alpha(u^\alpha) &= F^\alpha \\ \sigma^\alpha(u^\alpha) &= E^\alpha \varepsilon(u^\alpha) \\ \varepsilon(u^\alpha) &= \tfrac{1}{2}(\nabla u^\alpha + \nabla^\top u^\alpha) \end{aligned}$$

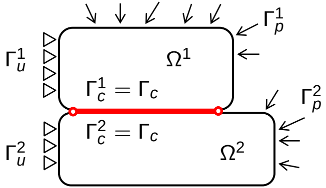
u^α ... displacement, σ^α ... stress
F^α ... volume force
E^α ... symmetric, elliptic elasticity tensor

- Dirichlet and Neumann b.c.:

$$\left.\begin{aligned} u^\alpha &= 0 \quad \text{on } \Gamma_u^\alpha \\ \sigma^\alpha(u^\alpha)\nu^\alpha &= P^\alpha \quad \text{on } \Gamma_P^\alpha \end{aligned}\right\} \quad \alpha = 1, 2$$

P^α ... surface traction
ν^α ... outer normal vector to $\partial\Omega^\alpha$

Contact conditions on Γ_C – Coulomb friction



- unilateral contact law:

$$u_\nu \le 0,\ \sigma_\nu \le 0,\ \sigma_\nu u_\nu = 0$$

where $u_\nu := u^1 \cdot \nu^1 + u^2 \cdot \nu^2$ and
$\sigma_\nu := \sigma^1(u^1)\nu^1 \cdot \nu^1$

- transmission of contact stresses:

$$\sigma^1(u^1)\nu^1 + \sigma^2(u^2)\nu^2 = 0$$

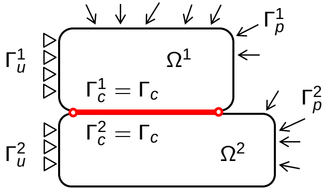
- Coulomb friction law:

$$\begin{array}{lll} |\sigma_t| \le -\mathcal{F}\sigma_\nu & & \\ |\sigma_t| < -\mathcal{F}\sigma_\nu & \Rightarrow & u_t = 0 \\ |\sigma_t| = -\mathcal{F}\sigma_\nu & \Rightarrow & \exists\kappa \ge 0 :\ u_t = -\kappa\sigma_t \end{array}$$

where $u_t := u^1 \cdot t^1 + u^2 \cdot t^2$,
$\sigma_t := \sigma^1(u^1)\nu^1 \cdot t^1$,
and t^α is a tangential vector to ν^α

- $\mathcal{F} \ge 0$ is the coefficient of friction

Contact conditions on Γ_C – Tresca friction



- unilateral contact law:

$$u_\nu \leq 0,\ \sigma_\nu \leq 0,\ \sigma_\nu u_\nu = 0$$

where $u_\nu := u^1 \cdot \nu^1 + u^2 \cdot \nu^2$ and $\sigma_\nu := \sigma^1(u^1)\nu^1 \cdot \nu^1$

- transmission of contact stresses:

$$\sigma^1(u^1)\nu^1 + \sigma^2(u^2)\nu^2 = 0$$

- Tresca friction law:

$$\begin{array}{lll} |\sigma_t| \leq g & & \\ |\sigma_t| < g & \Rightarrow & u_t = 0 \\ |\sigma_t| = g & \Rightarrow & \exists \kappa \geq 0 : \ u_t = -\kappa \sigma_t \end{array}$$

where $g \geq 0$ is an à-priori given slip bound

Fixed point idea - Coulomb friction

- auxiliary problem with the Tresca friction has the unique solution
- it defines the mapping
$$\Psi : g \mapsto -\mathcal{F}\sigma_\nu(u(g))$$
- fixed point of this mapping, i.e. g such that
$$\Psi(g) = g,$$
defines the solution $u = u(g)$ to the contact problem with the Coulomb friction
- the method of successive approximations is the natural tool for computations:
$$\textit{Initialize } \; g^{(0)} : \quad g^{(k+1)} = \Psi(g^{(k)}), \quad k = 0, 1, 2, \ldots$$

Weak formulation – Tresca friction

- Sets:

$$\begin{aligned}
\mathbb{V}^\alpha &= \{v^\alpha \in (H^1(\Omega^\alpha))^2 |\, v^\alpha = 0 \text{ on } \Gamma_u^\alpha\}, \quad \alpha = 1,2 \\
\mathbb{V} &= \mathbb{V}^1 \times \mathbb{V}^2, \quad v = (v^1, v^2) \in \mathbb{V} \\
\mathbb{K} &= \{v \in \mathbb{V} |\, v_\nu \le 0 \text{ on } \Gamma_C\} \\
\mathbb{K}^2(\varphi) &= \{v^2 \in \mathbb{V}^2 |\, v^2 \cdot \nu^2 \le -\varphi \text{ on } \Gamma_C\}, \quad \varphi \in L^2(\Gamma_C) \\
\mathbb{H}^{1/2}(\Gamma_C) &= \{\varphi \in (L^2(\Gamma_C))^2 |\, \exists v^\alpha \in \mathbb{V}^\alpha : \varphi = v^\alpha \text{ on } \Gamma_C\} \\
\mathbb{V}_0^\alpha &= \{v^\alpha \in (H^1(\Omega^\alpha))^2 |\, v^\alpha = 0 \text{ on } \Gamma_u^\alpha \cap \Gamma_C\}, \quad \alpha = 1,2
\end{aligned}$$

- Smoothness assumptions: $F^\alpha \in (L^2(\Omega^\alpha))^2$, $P^\alpha \in (L^2(\Gamma_p^\alpha))^2$, $\alpha = 1, 2$, $g \in L^2(\Gamma_C)$, $g \ge 0$, and $\Gamma_u^1 \cap \Gamma_C = \Gamma_u^2 \cap \Gamma_C$

Weak formulation – Tresca problem

$$(\mathcal{P}) \quad \begin{cases} \textit{Find } u \in \mathbb{K} \textit{ such that} \\ a(u, v-u) + j(v) - j(u) \geq b(v-u) \quad \forall v \in \mathbb{K}, \end{cases}$$

where

$$a : \mathbb{V} \times \mathbb{V} \mapsto \mathbb{R}, \quad a(u,v) = a^1(u^1, v^1) + a^2(u^2, v^2),$$

$$b : \mathbb{V} \mapsto \mathbb{R}, \quad b(v) = b^1(v^1) + b^2(v^2),$$

$$j : \mathbb{V} \mapsto \mathbb{R}, \quad j(v) = j(v^1, v^2) = \int_{\Gamma_C} g|v_t|\, ds,$$

and

$$a^\alpha(u^\alpha, v^\alpha) = \int_{\Omega^\alpha} E^\alpha \varepsilon(u^\alpha) : \varepsilon(v^\alpha)\, dx, \quad b^\alpha(v^\alpha) = \int_{\Omega^\alpha} F^\alpha \cdot v^\alpha\, dx + \int_{\Gamma_P^\alpha} P^\alpha \cdot v^\alpha\, ds, \quad \alpha = 1, 2.$$

Splitting idea

Proposition A pair $u = (u^1, u^2) \in \mathbb{V}$ is a solution to $(\mathcal{P})$ iff u^1 and u^2 solve the following coupled problems:

$$(\mathcal{P}_1) \quad \begin{cases} \text{Find } u^1 \in \mathbb{V}^1 \text{ such that} \\ a^1(u^1, v^1) = b^1(v^1) - a^2(u^2, \pi^2 v^1) + b^2(\pi^2 v^1) \quad \forall v^1 \in \mathbb{V}^1, \end{cases}$$

$$(\mathcal{P}_2) \quad \begin{cases} \text{Find } u^2 \in \mathbb{K}^2(u^1 \cdot \nu^1) \text{ such that} \\ a^2(u^2, v^2 - u^2) + j(u^1, v^2) - j(u^1, u^2) \geq b^2(v^2 - u^2) \quad \forall v^2 \in \mathbb{K}^2(u^1 \cdot \nu^1), \end{cases}$$

where the extension mappings $\pi^\alpha : \mathbb{H}^{1/2}(\Gamma_C) \mapsto \mathbb{V}^\alpha$ are defined for $\lambda \in \mathbb{H}^{1/2}(\Gamma_C)$ by:

$$\begin{cases} \pi^\alpha \lambda \in \mathbb{V}^\alpha \quad : \quad a^\alpha(\pi^\alpha \lambda, v^\alpha) = 0 \quad \forall v^\alpha \in \mathbb{V}^\alpha_0 \\ \qquad\qquad\qquad\quad \pi^\alpha \lambda = \lambda \text{ on } \Gamma_C \end{cases}$$

Outline

Four auxiliary subproblems

- consider $\lambda \in \mathbb{H}^{1/2}(\Gamma_C)$ that plays the role of the contact displacemnt
- linear elasticity for Ω^1 with the prescribed displacements λ on Γ_C

$$(\mathcal{P}_1(\lambda)) \quad \begin{cases} \textit{Find } u^1 := u^1(\lambda) \in \mathbb{V}^1 \textit{ such that} \\ a^1(u^1, v^1) = b^1(v^1) \quad \forall v^1 \in \mathbb{V}^1_0 \\ u^1 = \lambda \quad \text{on } \Gamma_C \end{cases}$$

- Tresca problem for Ω^2 with the prescribed gap λ on Γ_C

$$(\mathcal{P}_2(\lambda)) \quad \begin{cases} \textit{Find } u^2 := u^2(\lambda) \in \mathbb{K}^2(\lambda \cdot \nu^1) \textit{ such that} \\ a^2(u^2, v^2 - u^2) + j(\lambda, v^2) - j(\lambda, u^2) \geq b^2(v^2 - u^2) \quad \forall v^2 \in \mathbb{K}^2(\lambda \cdot \nu^1) \end{cases}$$

Four auxiliary problems

- Neumann problem on Γ_C for Ω^1

$$(\mathcal{P}_3(\lambda)) \quad \left\{ \begin{array}{l} \textit{Find } w^1 := w^1(\lambda) \in \mathbb{V}^1 \textit{ such that} \\ a^1(w^1, v^1) = \frac{1}{2}(-a^1(u^1, v^1) + b^1(v^1) - a^2(u^2(\lambda), \pi^2 v^1) + b^2(\pi^2 v^1)) \\ \hfill \forall v^1 \in \mathbb{V}^1 \end{array} \right.$$

- Neumann problem on Γ_C for Ω^2

$$(\mathcal{P}_4(\lambda)) \quad \left\{ \begin{array}{l} \textit{Find } w^2 := w^2(\lambda) \in \mathbb{V}^2 \textit{ such that} \\ a^2(w^2, v^2) = \frac{1}{2}(a^2(u^2, v^2) - b^2(v^2) + a^1(u^1(\lambda), \pi^1 v^2) - b^1(\pi^1 v^2)) \\ \hfill \forall v^2 \in \mathbb{V}^2 \end{array} \right.$$

ALGORITHM (DD)

Let $\lambda_0 \in \mathbb{H}^{1/2}(\Gamma_C)$ and $\theta > 0$ be given.

For $k \geq 1$ integer, define u_k^α, w_k^α, $\alpha = 1, 2$ and λ_k by:

Step 1. $u_k^1 \in \mathbb{V}^1$ solves $(\mathcal{P}_1(\lambda_{k-1}))$

Step 2. $u_k^2 \in \mathbb{K}^2(\lambda_{k-1} \cdot \nu^1)$ solves $(\mathcal{P}_2(\lambda_{k-1}))$

Step 3. $w_k^1 \in \mathbb{V}^1$ solves $(\mathcal{P}_3(\lambda_{k-1}))$

Step 4. $w_k^2 \in \mathbb{V}^2$ solves $(\mathcal{P}_4(\lambda_{k-1}))$

Step 5. $\lambda_k = \lambda_{k-1} + \theta(w_k^1 - w_k^2)$ on Γ_C

- choice of the update parameter θ crucial for the convergence

Convergence result

Theorem There exist: $0 < \theta^* < 4$ and functions $\lambda_* \in \mathbb{H}^{1/2}(\Gamma_C)$, $u_*^\alpha, w_*^\alpha \in \mathbb{V}^\alpha$, $\alpha = 1, 2$ such that for any $\theta \in (0, \theta^*)$ it follows that

$$\begin{aligned} &\lambda_k \to \lambda_* && \text{in } \mathbb{H}^{1/2}(\Gamma_C), \\ &\left.\begin{aligned} u_k^\alpha &\to u_*^\alpha \\ w_k^\alpha &\to w_*^\alpha \end{aligned}\right\} && \text{in } (H^1(\Omega^\alpha))^2,\ \alpha = 1, 2, \end{aligned}$$

where the sequence $\{(u_k^\alpha, w_k^\alpha, \lambda_k)\}$ is generated by ALGORITHM (**DD**). In addition, the couple (u_*^1, u_*^2) solves $(\mathcal{P})$.

Proof is based on $T_\theta : \mathbb{H}^{1/2}(\Gamma_C) \mapsto \mathbb{H}^{1/2}(\Gamma_C)$, $T_\theta\lambda = \lambda + \theta(w^1(\lambda) - w^2(\lambda))$, satisfying

$$|||T_\theta\lambda - T_\theta\tilde{\lambda}|||_{\mathbb{H}^{1/2}(\Gamma_C)} \le (1 + c^2\theta^2 - \theta)|||\lambda - \tilde{\lambda}|||_{\mathbb{H}^{1/2}(\Gamma_C)}$$

that is contractive if $\theta < \theta^* := 1/c^2$. Moreover, since $c > 1/2$, we get $\theta^* < 4$.

Outline

FE approximation of Tresca problem

$$(\mathcal{P})_h \qquad \begin{cases} \textit{Find } u_h \in \mathbb{K}_h \textit{ such that} \\ a(u_h, v_h - u_h) + j(v_h) - j(u_h) \geq b(v_h - u_h) \quad \forall v_h \in \mathbb{K}_h, \end{cases}$$

where

$\mathbb{V}_h$ is piecewise linear approximation of $\mathbb{V}$ with the mesh-norm h,

$\mathbb{K}_h \approx \mathbb{K}, \quad \mathbb{K}_h = \{v_h \in \mathbb{V}_h |\, v_{h\nu} \leq 0 \;\text{ on }\; \Gamma_C\},$

$\mathcal{W}_h \approx \mathbb{H}^{1/2}(\Gamma_C),$

$\mathbb{V}_h^1 \approx \mathbb{V}^1,\ \mathbb{V}_h^2 \approx \mathbb{V}^2,$

$\mathbb{K}_h^2(\lambda_h \cdot \nu^1) \approx \mathbb{K}^2(\lambda_h \cdot \nu^1)$ for $\lambda_h \in \mathcal{W}_h$.

ALGORITHM $(\mathbf{DD})_h$

Let $\lambda_{0,h} \in \mathcal{W}_h$ and $\theta > 0$ be given.

For $k \geq 1$ integer, define $u^\alpha_{k,h}$, $w^\alpha_{k,h}$, $\alpha = 1, 2$ and $\lambda_{k,h}$ by:

Step 1 $u^1_{k,h} \in \mathbb{V}^1_h$ solves $(\mathcal{P}_1(\lambda_{k-1,h}))_h$

Step 2 $u^2_{k,h} \in \mathbb{K}^2_h(\lambda_{k-1,h} \cdot \nu^1)$ solves $(\mathcal{P}_2(\lambda_{k-1,h}))_h$

Step 3 $w^1_{k,h} \in \mathbb{V}^1_h$ solves $(\mathcal{P}_3(\lambda_{k-1,h}))_h$

Step 4 $w^2_{k,h} \in \mathbb{V}^2_h$ solves $(\mathcal{P}_4(\lambda_{k-1,h}))_h$

Step 5 $\lambda_{k,h} = \lambda_{k-1,h} + \theta(w^1_{k,h} - w^2_{k,h})$ on Γ_C

Convergence result

Theorem There exist: $0 < \theta^* < 4$ which does not depend on $h > 0$ and functions $\lambda_{*,h} \in \mathcal{W}_h$, $u^\alpha_{*,h}, w^\alpha_{*,h} \in \mathbb{V}^\alpha_h$, $\alpha = 1, 2$ such that for any $\theta \in (0, \theta^*)$ it follows:

$$\left.\begin{array}{l} \lambda_{k,h} \to \lambda_{*,h}, \\ u^\alpha_{k,h} \to u^\alpha_{*,h}, \\ w^\alpha_{k,h} \to w^\alpha_{*,h}, \end{array}\right\} \quad k \to \infty, \ \alpha = 1, 2,$$

where the sequence $\{(u^\alpha_{k,h}, w^\alpha_{k,h}, \lambda_{k,h})\}$ is generated by ALGORITHM (**DD**)$_h$. In addition, the couple $(u^1_{*,h}, u^2_{*,h})$ solves $(\mathcal{P})_h$.

Proof is analogous to the continuous setting. The constant c is independent of h.

Outline

Matrix formulation

FEM $\Longrightarrow$ primal representation

- $\mathbf{A}_\alpha \in \mathbb{R}^{n_\alpha \times n_\alpha}$ symmetric, positive definite stiffness matrices on Ω^α
- $\mathbf{b}_\alpha \in \mathbb{R}^{n_\alpha}$ corresponding load vectors
- $\mathbf{N}_\alpha, \mathbf{T}_\alpha \in \mathbb{R}^{m \times n_\alpha}$ matrices given by normal and tangential unit vectors ν^α, t^α at contact nodes, respectively, and

$$\mathbf{B}_\alpha = \begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{T}_\alpha \end{pmatrix}, \quad \alpha = 1, 2$$

- $\mathbf{g} \in \mathbb{R}^m$ vector of slip bound values g_i at contact nodes

Dual representation

- $\mathbf{C}_\alpha := \mathbf{B}_\alpha \mathbf{A}_\alpha^{-1} \mathbf{B}_\alpha^\top \in \mathbb{R}^{2m \times 2m}$ symmetric, positive definite Schur complements
- $\mathbf{d}_\alpha := \mathbf{B}_\alpha \mathbf{A}_\alpha^{-1} \mathbf{b}_\alpha + (-1)^\alpha \boldsymbol{\lambda}_{k-1}, \quad \alpha = 1, 2$

ALGORITHM $(\mathbf{DD})_h$ – matrix formulation

Let $\boldsymbol{\lambda}_0 \in \mathbb{R}^{2m}$ and $\theta > 0$ be given. For $k \geq 1$, compute $\mathbf{u}_{\alpha,k}, \mathbf{w}_{\alpha,k} \in \mathbb{R}^{n_\alpha}$, $\alpha = 1, 2$ and $\boldsymbol{\lambda}_k \in \mathbb{R}^{2m}$ by:

<u>*Step 1*</u> $\mathbf{u}_{1,k} := \operatorname{argmin} \frac{1}{2}\mathbf{v}_1^\top \mathbf{A}_1 \mathbf{v}_1 - \mathbf{v}_1^\top \mathbf{b}_1, \quad \text{s.t. } \mathbf{B}_1 \mathbf{v}_1 = \boldsymbol{\lambda}_{k-1}$

<u>*Step 2*</u> $\mathbf{u}_{2,k} := \operatorname{argmin} \frac{1}{2}\mathbf{v}_2^\top \mathbf{A}_2 \mathbf{v}_2 - \mathbf{v}_2^\top \mathbf{b}_2 + \sum_{i=1}^m g_i |\boldsymbol{\lambda}_{t,k-1} + \mathbf{T}_2 \mathbf{v}_2|_i$

$\text{s.t. } \boldsymbol{\lambda}_{\nu,k-1} + \mathbf{N}_2 \mathbf{v}_2 \leq \mathbf{0}$

<u>*Step 3*</u> $\mathbf{A}_1 \mathbf{w}_{1,k} = \frac{1}{2}\mathbf{B}_1^\top (\mathbf{B}_1(\mathbf{b}_1 - \mathbf{A}_1 \mathbf{u}_{1,k}) - \mathbf{B}_2(\mathbf{b}_2 - \mathbf{A}_2 \mathbf{u}_{2,k}))$

<u>*Step 4*</u> $\mathbf{A}_2 \mathbf{w}_{2,k} = \frac{1}{2}\mathbf{B}_2^\top (\mathbf{B}_1(\mathbf{b}_1 - \mathbf{A}_1 \mathbf{u}_{1,k}) - \mathbf{B}_2(\mathbf{b}_2 - \mathbf{A}_2 \mathbf{u}_{2,k}))$

<u>*Step 5*</u> $\boldsymbol{\lambda}_k = \boldsymbol{\lambda}_{k-1} + \theta(\mathbf{B}_1 \mathbf{w}_{1,k} + \mathbf{B}_2 \mathbf{w}_{2,k})$

Schur complement in Step 1

- *Step 1* is equivalent to the saddle-point linear system:

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1^\top \\ \mathbf{B}_1 & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{1,k} \\ \mathbf{s}_{1,k} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \boldsymbol{\lambda}_{k-1} \end{pmatrix}$$

- eliminating the first unknown we get the linear system for the Lagrange multiplier:

$$\mathbf{C}_1 \mathbf{s}_{1,k} = \mathbf{d}_1$$

- it can be solved by the CGM

Schur complement in Step 2

- *Step 2* is equivalent to the saddle-point problem:

$$\left.\begin{array}{l} \textit{Find } (\mathbf{u}_{2,k}, \mathbf{s}_{2,k}) \in \mathbb{R}^{n_2} \times \Lambda_2(\mathbf{g}) \textit{ such that} \\ \mathcal{L}_2(\mathbf{u}_{2,k}, \mathbf{s}_2) \le \mathcal{L}_2(\mathbf{u}_{2,k}, \mathbf{s}_{2,k}) \le \mathcal{L}_2(\mathbf{v}_2, \mathbf{s}_{2,k}) \quad \forall (\mathbf{v}_2, \mathbf{s}_2) \in \mathbb{R}^{n_2} \times \Lambda_2(\mathbf{g}), \end{array}\right\}$$

  where $\mathcal{L}_2(\mathbf{v}_2, \mathbf{s}_2) := \frac{1}{2}\mathbf{v}_2^\top \mathbf{A}_2 \mathbf{v}_2 - \mathbf{v}_2^\top \mathbf{b}_2 + \mathbf{s}_2^\top (\mathbf{B}_2 \mathbf{v}_2 + \boldsymbol{\lambda}_{k-1})$ and

$$\Lambda_2(\mathbf{g}) := \{\mathbf{s}_2 = (\mathbf{s}_{2,\nu}^\top, \mathbf{s}_{2,t}^\top)^\top \in \mathbb{R}^{2m} : \mathbf{s}_{2,\nu} \ge \mathbf{0},\ |\mathbf{s}_{2,t}| \le \mathbf{g}\}$$

- $\mathbf{s}_{2,\nu}$ treats the inequality constraint, $\mathbf{s}_{2,t}$ regularizes the non-differentiable term:

$$\sum_{i=1}^{m} g_i |\boldsymbol{\lambda}_{t,k-1} + \mathbf{T}_2 \mathbf{v}_2|_i = \max_{|\mathbf{s}_{2,t}| \le \mathbf{g}} (\boldsymbol{\lambda}_{t,k-1} + \mathbf{T}_2 \mathbf{v}_2)^\top \mathbf{s}_{2,t}$$

- eliminating the first unknown we get:

$$\mathbf{s}_{2,k} := \operatorname{argmin} \frac{1}{2} \mathbf{s}_2^\top \mathbf{C}_2 \mathbf{s}_2 - \mathbf{s}_2^\top \mathbf{d}_2, \quad \text{s.t.} \quad \mathbf{s}_{2,\nu} \ge \mathbf{0},\ |\mathbf{s}_{2,t}| \le \mathbf{g}$$

- we use the active set algorithm combining the CGM with the gradient projections (Dostál, Schőberl, 2005)

ALGORITHM $(\mathbf{DD})_h$ – dual matrix form

Let $\boldsymbol{\lambda}_0 \in \mathbb{R}^{2m}$ and $\theta > 0$ be given. For $k \geq 1$, compute $\mathbf{s}_{\alpha,k} \in \mathbb{R}^{2m}$, $\mathbf{w}_{\alpha,k} \in \mathbb{R}^{n_\alpha}$, $\alpha = 1, 2$ and $\boldsymbol{\lambda}_k \in \mathbb{R}^{2m}$ by:

Step 1 $\quad \mathbf{C}_1 \mathbf{s}_{1,k} = \mathbf{d}_1$

Step 2 $\quad \mathbf{s}_{2,k} := \operatorname{argmin} \frac{1}{2} \mathbf{s}_2^\top \mathbf{C}_2 \mathbf{s}_2 - \mathbf{s}_2^\top \mathbf{d}_2, \quad \text{s.t.} \quad \mathbf{s}_{2,\nu} \geq \mathbf{0}, \ |\mathbf{s}_{2,t}| \leq \mathbf{g}$

Step 3 $\quad \mathbf{A}_1 \mathbf{w}_{1,k} = \frac{1}{2} \mathbf{B}_1^\top (\mathbf{s}_{1,k} - \mathbf{s}_{2,k})$

Step 4 $\quad \mathbf{A}_2 \mathbf{w}_{2,k} = \frac{1}{2} \mathbf{B}_2^\top (\mathbf{s}_{1,k} - \mathbf{s}_{2,k})$

Step 5 $\quad \boldsymbol{\lambda}_k = \boldsymbol{\lambda}_{k-1} + \theta(\mathbf{B}_1 \mathbf{w}_{1,k} + \mathbf{B}_2 \mathbf{w}_{2,k})$

- displacements $\mathbf{u}_1, \mathbf{u}_2$ are computed after terminating the algorithm

Modified ALGORITHM (DD)$_h$

Let $\boldsymbol{\lambda}_0 \in \mathbb{R}^{2m}$ and $\theta > 0$ be given. For $k \geq 1$, compute $\mathbf{s}_{\alpha,k} \in \mathbb{R}^{2m}, \mathbf{w}_{\alpha,k} \in \mathbb{R}^{n_\alpha}$, $\alpha = 1, 2$ and $\boldsymbol{\lambda}_k \in \mathbb{R}^{2m}$ by:

Step 1 $\quad \mathbf{s}_{1,k} := \operatorname{argmin} \frac{1}{2} \mathbf{s}_1^\top \mathbf{C}_1 \mathbf{s}_1 - \mathbf{s}_1^\top \mathbf{d}_1, \quad \text{s.t.} \quad |\mathbf{s}_{1,t}| \leq \mathbf{g}$

Step 2 $\quad \mathbf{s}_{2,k} := \operatorname{argmin} \frac{1}{2} \mathbf{s}_2^\top \mathbf{C}_2 \mathbf{s}_2 - \mathbf{s}_2^\top \mathbf{d}_2, \quad \text{s.t.} \quad \mathbf{s}_{2,\nu} \geq \mathbf{0}$

Step 3 $\quad \mathbf{A}_1 \mathbf{w}_{1,k} = \frac{1}{2} \mathbf{B}_1^\top (\mathbf{s}_{1,k} - \mathbf{s}_{2,k})$

Step 4 $\quad \mathbf{A}_2 \mathbf{w}_{2,k} = \frac{1}{2} \mathbf{B}_2^\top (\mathbf{s}_{1,k} - \mathbf{s}_{2,k})$

Step 5 $\quad \boldsymbol{\lambda}_k = \boldsymbol{\lambda}_{k-1} + \theta(\mathbf{B}_1 \mathbf{w}_{1,k} + \mathbf{B}_2 \mathbf{w}_{2,k})$

- *Step 1*: Tresca friction and the prescribed normal displacement on Γ_c
- *Step 2*: unilateral condition and the prescribed tangential displacement on Γ_c

Further modifications

- the Gauss-Seidel splitting in *Step 1* and *Step 2* on the normal and tangential components reduces the size of the subproblems:

$$\mathbf{C}_\alpha = \begin{pmatrix} \mathbf{C}_{\alpha,\,\nu\nu} & \mathbf{C}_{\alpha,\,\nu t} \\ \mathbf{C}_{\alpha,\,t\nu} & \mathbf{C}_{\alpha,\,tt} \end{pmatrix}, \quad \mathbf{d}_\alpha = \begin{pmatrix} \mathbf{d}_{\alpha,\,\nu} \\ \mathbf{d}_{\alpha,\,t} \end{pmatrix}, \quad \alpha = 1, 2$$

- therefore, we get four splitting variants of each algorithm
- the precision control for inner solvers is adaptive (with respect to the outer accuracy)

Extension to the Coulomb friction

- inexact succesive approximations so that $\mathbf{g} := \mathbf{g}_k$ is updated:

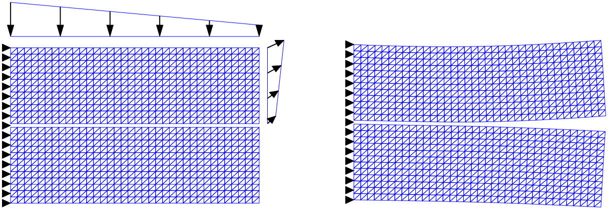
$$\mathbf{g}_k = \mathbf{s}_{2,\,k-1,\,\nu} \quad (\geq \mathbf{0})$$

- in other words, one iteration of a variant of ALGORITHM $(\mathbf{DD})_h$ is performed in each succesive iteration for computing the fixed point

Outline

Model problem

- Geometry: for n, m DOFs with $\theta = 0.2$



- outer iterations: $\bar{k}$
- matrix-vector multiplications: $n_{\mathbf{A}}$
- CPU time [sec.]: *time*
- relative efficiency: $eff := n_{\mathbf{A}}/(2m)$

Original algorithm – Tresca friction

$n\|m$	*without* $\bar{k}\|n_{\mathbf{A}}$ [$time\|eff$]	*in Step 1* $\bar{k}\|n_{\mathbf{A}}$ [$time\|eff$]	*in Step 2* $\bar{k}\|n_{\mathbf{A}}$ [$time\|eff$]	*in Step 1+2* $\bar{k}\|n_{\mathbf{A}}$ [$time\|eff$]
504|18	**46**|505 [0.17|14.03]	**47**|704 [0.17|19.56]	**46**|681 [0.17|18.92]	**48**|901 [0.22|25.03]
6072|66	**47**|565 [3.96|4.28]	**48**|759 [5.21|5.75]	**47**|785 [5.52|5.95]	**48**|972 [7.04|7.36]
17784|114	**47**|594 [20.48|2.61]	**48**|765 [26.30|3.36]	**47**|807 [27.50|3.54]	**48**|981 [33.42|4.30]
35640|162	**47**|604 [56.21|1.86]	**48**|798 [74.16|2.46]	**47**|827 [77.08|2.55]	**48**|1009 [94.07|3.11]
59640|210	**47**|591 [118.61|1.41]	**48**|798 [162.43|1.90]	**47**|837 [167.72|1.99]	**48**|1024 [204.02|2.44]
89784|258	**47**|604 [222.11|1.17]	**48**|809 [296.82|1.57]	**47**|852 [312.38|1.65]	**48**|1045 [382.92|2.03]

Modified algorithm – Tresca friction

$n \vert m$	*without* $\bar{k} \vert n_{\mathbf{A}}$ [*time* | *eff*]	*in Step 1* $\bar{k} \vert n_{\mathbf{A}}$ [*time* | *eff*]	*in Step 2* $\bar{k} \vert n_{\mathbf{A}}$ [*time* | *eff*]	*in Step 1+2* $\bar{k} \vert n_{\mathbf{A}}$ [*time* | *eff*]
504|18	**37**|405 [0.14|11.25]	**38**|572 [0.14|15.89]	**38**|575 [0.14|15.97]	**38**|729 [0.17|20.25]
6072|66	**36**|440 [3.07|3.33]	**38**|651 [4.49|4.93]	**37**|615 [4.41|4.66]	**38**|805 [5.34|6.10]
17784|114	**37**|544 [18.50|2.39]	**38**|658 [22.39|2.89]	**36**|744 [25.18|3.26]	**38**|846 [28.45|3.71]
35640|162	**37**|598 [55.24|1.85]	**38**|698 [64.37|2.15]	**36**|727 [66.97|2.24]	**38**|859 [78.97|2.65]
59640|210	**36**|618 [121.84|1.47]	**38**|735 [145.02|1.75]	**36**|799 [157.22|1.90]	**38**|877 [172.91|2.09]
89784|258	**37**|660 [245.56|1.28]	**38**|726 [279.49|1.41]	**36**|787 [290.58|1.53]	**38**|876 [321.56|1.70]

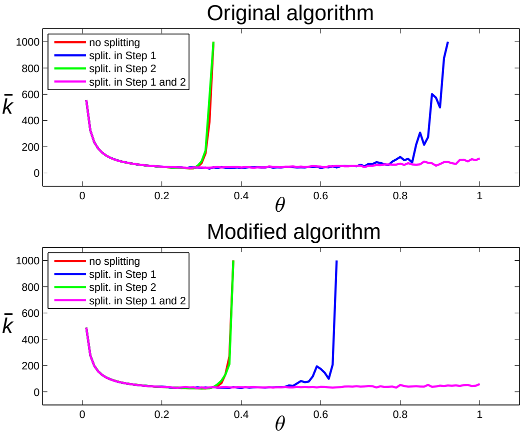
Original algorithm – Coulomb friction

$n \| m$	without $\bar{k} \| n_{\mathbf{A}}$ [*time*|*eff*]	in Step 1 $\bar{k} \| n_{\mathbf{A}}$ [*time*|*eff*]	in Step 2 $\bar{k} \| n_{\mathbf{A}}$ [*time*|*eff*]	in Step 1+2 $\bar{k} \| n_{\mathbf{A}}$ [*time*|*eff*]
504|18	**60**|667 [0.80|18.53]	**61**|1075 [0.98|29.86]	**59**|742 [0.67|20.61]	**60**|1146 [0.70|31.83]
6072|66	**61**|1044 [8.19|7.91]	**61**|1492 [8.35|11.30]	**61**|824 [4.63|6.24]	**60**|1236 [6.91|9.36]
17784|114	**62**|1313 [31.73|5.76]	**63**|1816 [43.71|7.96]	**61**|855 [33.24|3.75]	**63**|1365 [32.89|5.99]
35640|162	**61**|1839 [126.94|5.68]	**62**|1819 [133.30|5.61]	**61**|892 [59.59|2.75]	**62**|1377 [91.82|4.25]
59640|210	**60**|1583 [238.32|3.77]	**61**|2336 [341.33|5.56]	**61**|876 [127.42|2.09]	**61**|1377 [196.11|3.28]
89784|258	**60**|1627 [405.31|3.15]	**59**|2333 [585.25|4.52]	**60**|864 [216.09|1.67]	**61**|1421 [359.08|2.75]

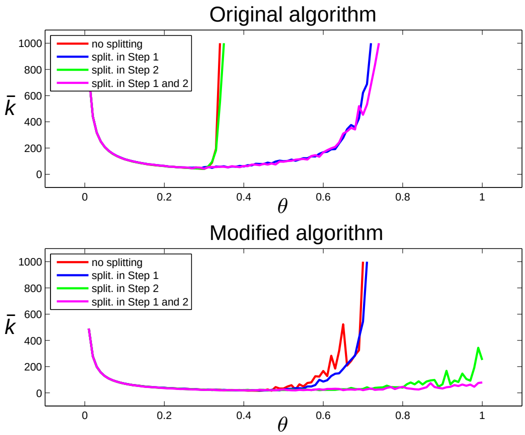
Modified algorithm – Coulomb friction

$n \| m$	*without* $\bar{k} \| n_{\mathbf{A}}$ [*time* | *eff*]	*in Step 1* $\bar{k} \| n_{\mathbf{A}}$ [*time* | *eff*]	*in Step 2* $\bar{k} \| n_{\mathbf{A}}$ [*time* | *eff*]	*in Step 1+2* $\bar{k} \| n_{\mathbf{A}}$ [*time* | *eff*]
504|18	**37**|530 [0.19|14.72]	**36**|520 [0.16|14.44]	**37**|714 [0.19|19.83]	**38**|770 [0.19|21.39]
6072|66	**36**|987 [5.76|7.48]	**37**|586 [3.29|4.44]	**37**|964 [5.35|7.30]	**38**|829 [4.59|6.28]
17784|114	**36**|1417 [34.32|6.21]	**38**|626 [15.16|2.75]	**37**|1347 [32.81|5.91]	**35**|794 [19.00|3.48]
35640|162	**37**|1864 [119.50|5.75]	**36**|608 [38.74|1.88]	**36**|1399 [89.79|4.32]	**36**|863 [54.83|2.66]
59640|210	**37**|2132 [290.71|5.08]	**37**|624 [93.40|1.49]	**37**|1401 [191.30|3.34]	**35**|851 [115.64|2.03]
89784|258	**37**|2532 [631.80|4.91]	**37**|619 [154.52|1.20]	**37**|1806 [451.65|3.50]	**36**|877 [225.59|1.70]

Sensitivity with respect to θ – Tresca friction



Sensitivity with respect to θ – Coulomb friction



Outline

Conclusions

- algorithm decomposes the contact problem on the seperate bodies
- bound on the update parameter θ is independent of h
- number of outer iterations is independent of h
- number of matrix-vector multiplications increases slightly with the size of the problem
- but, so that the relative efficiency decreases
- the experimental observation is that the splitting (of contact conditions or of the Gauss-Seidel type) stabilizes computations, i.e., the algorithm converges for larger values of θ

Reference

- Haslinger, J., Kučera, R., Riton, J., Sassi, T.: A domain decomposition method for two-body contact problems with Tresca friction. Submitted to Advances in Computational Mathematics (2012).

Thank You for Your Attention!