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INVESTMENTS IN EDUCATION DEVELOPMENT

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Lecture 2 : Domain Decomposition Algorithms : Convergence results

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Outline

Interface Equation

We consider the model problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

For a nonoverlapping domain decomposition, we can write the equivalent multidomain formulation

$$\begin{cases} -\Delta u_1 = f & \text{in } \Omega_1 \\ u_1 = 0 & \text{on } \partial\Omega_1 \backslash \Gamma \\ -\Delta u_2 = f & \text{in } \Omega_2 \\ u_2 = 0 & \text{on } \partial\Omega_2 \backslash \Gamma \\ u_1 = u_2 & \text{on } \Gamma \\ \dfrac{\partial u_1}{\partial n_1} = -\dfrac{\partial u_2}{\partial n_2} & \text{on } \Gamma \end{cases}$$

Let λ be the unknown value of the solution u on the interface Γ : $\lambda = u|_\Gamma$. Should we know a priori the value λ on Γ, we could solve the following two independent boundary-value problems with Dirichlet condition on Γ $(k = 1, 2)$:

$$\begin{cases} -\Delta w_k = f & \text{in } \Omega_k \\ w_k = 0 & \text{on } \partial\Omega_k \backslash \Gamma \\ w_k = \lambda & \text{on } \Gamma \end{cases}$$

Let us note that the solutions $w_k(f, \lambda)$, $k = 1, 2$, of the above problems depend on f and λ.

To obtain the value λ on Γ, let us split w_k as follows

$$w_k = w_k(f, 0) + w_k(0, \lambda)$$

where $w_k(f, 0)$ and $w_k(0, \lambda)$ are respectively the solutions of the following problems

$$\begin{cases} -\Delta w_k = f & \text{in } \Omega_k \\ w_k = 0 & \text{on } \partial\Omega_k \backslash \Gamma \\ w_k = 0 & \text{on } \Gamma \end{cases}$$

$$\begin{cases} -\Delta w_k = 0 & \text{in } \Omega_k \\ w_k = 0 & \text{on } \partial\Omega_k \backslash \Gamma \\ w_k = \lambda & \text{on } \Gamma \end{cases}$$

- The function $w_k(f,0)$ depend only on the source data f then $w_k(f,0) = Q_k f$ where Q_k is a linear continuous operator.
- The solution $w_k(0,\lambda)$ depend only on the value λ on Γ then $w_k(0,\lambda) = R_k\lambda$, where R_k is the harmonic extention operator of the trace λ on Ω_k. Using

$$\frac{\partial w_1}{\partial n} = \frac{\partial w_2}{\partial n} \qquad \text{on } \Gamma$$

we have

$$\frac{\partial}{\partial n}(Q_1 f + R_1\lambda) = \frac{\partial}{\partial n}(Q_2 f + R_2\lambda) \qquad \text{on } \Gamma$$

Therefore

$$\left(\frac{\partial R_1}{\partial n} - \frac{\partial R_2}{\partial n}\right)\lambda = \left(\frac{\partial Q_2}{\partial n} - \frac{\partial Q_1}{\partial n}\right)f \qquad \text{on } \Gamma$$

We have obtained an interface equation for unknown λ on Γ

$$S\lambda = L \text{ on } \Gamma,$$

defined by the Steklov-Poincaré operator :

$$S\mu = (S_1 + S_2)\mu = \sum_{k=1}^{2} \frac{\partial}{\partial n_k} R_k \mu$$

- For $k = 1, 2$ the Steklov-Poincaré operator S_k maps any trace function μ on Γ onto its the normal derivative

$$\begin{aligned} S_k : H_{00}^{\frac{1}{2}}(\Gamma) &\longrightarrow (H_{00}^{\frac{1}{2}}(\Gamma))' \\ \mu &\longmapsto \frac{\partial}{\partial n_k} R_k \mu \end{aligned}$$

- For $k = 1, 2$ the Steklov-Poincaré operator S_k is linear, continuous and bijective
- L is a linear functional which depends on f :

$$L = \frac{\partial}{\partial n} Q_2 f - \frac{\partial}{\partial n} Q_1 f \qquad \text{on } \Gamma$$

starting from an initial guess $\hat{u}_2^0$ on Γ, for $m > 1$ solve the problems

$$\begin{cases} -\Delta {u_1}^m & = & f & \text{in } \Omega_1 \\ {u_1}^m & = & 0 & \text{on } \partial\Omega_1 \backslash \Gamma \\ {u_1}^m & = & \hat{u}_2^{m-1} & \text{on } \Gamma \end{cases}$$

Then solve

$$\begin{cases} -\Delta {u_2}^m & = & f & \text{in } \Omega_2 \\ {u_2}^m & = & 0 & \text{on } \partial\Omega_2 \backslash \Gamma \\ \dfrac{\partial {u_2}^m}{\partial n_2} & = & \dfrac{\partial {u_1}^m}{\partial n_1} & \text{on } \Gamma \end{cases}$$

$$\hat{u}_2^m = (1-\theta)\hat{u}_2^{m-1} + \theta {u_2}^m \text{ on } \Gamma$$

In order to decouple the influence of the source data f and the boundary data $\hat{u}_2^{m-1}$, we define the function $U_1 = u_1^m(0, f)$ as solution of

$$\begin{cases} -\Delta {u_1}^m &= f & \text{in } \Omega_1 \\ {u_1}^m &= 0 & \text{on } \partial\Omega_1 \backslash \Gamma \\ {u_1}^m &= 0 & \text{on } \Gamma \end{cases}$$

Moreover, we define the operator R_1 mapping any $\hat{u}_2^{m-1}$ on the trace u_1^m where u_1^m is the solution

$$\begin{cases} -\Delta {u_1}^m &= 0 & \text{in } \Omega_1 \\ {u_1}^m &= 0 & \text{on } \partial\Omega_1 \backslash \Gamma \\ {u_1}^m &= \hat{u}_2^{m-1} & \text{on } \Gamma \end{cases}$$

Then

$$u_1^m = U_1 + R_1(\hat{u}_2^{m-1})$$

In the same way, we define the function $U_2 = u_2^m(0, f)$ as solution

$$\begin{cases} -\Delta {u_2}^m & = & f & \text{in } \Omega_2 \\ {u_2}^m & = & 0 & \text{on } \partial\Omega_2 \backslash \Gamma \\ \dfrac{\partial {u_2}^m}{\partial n_2} & = & 0 & \text{on } \Gamma \end{cases}$$

and $u_2^m(\frac{\partial {u_1}^m}{\partial n_1}, 0)$ as solution of the Neumann problem

$$\begin{cases} -\Delta {u_2}^m & = & 0 & \text{in } \Omega_2 \\ {u_2}^m & = & 0 & \text{on } \partial\Omega_2 \backslash \Gamma \\ \dfrac{\partial {u_2}^m}{\partial n_2} & = & -\dfrac{\partial {u_1}^m}{\partial n_1} & \text{on } \Gamma \end{cases}$$

By the definition of the Steklov-Poincaré

$$\begin{aligned} {u_2}^m & = U_2 - S_2^{-1} \frac{\partial {u_1}^m}{\partial n_1} \\ & = U_2 - S_2^{-1} S_1 {u_1}^m \\ & = U_2 - S_2^{-1} S_1 R_1 \hat{u}_2^{m-1} \end{aligned}$$

Now the new iterate can be written in terms of a convex combination of the old one and the updated

$$\begin{aligned}\hat{u}_2^m &= (1-\theta)\hat{u}_2^{m-1} + \theta {u_2}^m \\ &= (1-\theta)\hat{u}_2^{m-1} + \theta\Big(U_2 - S_2^{-1}S_1R_1\hat{u}_2^{m-1}\Big) \\ &= \hat{u}_2^{m-1} - \theta\Big(\hat{u}_2^{m-1} + S_2^{-1}S_1R_1\hat{u}_2^{m-1} - U_2\Big) \\ &= T_\theta(\hat{u}_2^{m-1})\end{aligned}$$

where

$$\begin{aligned}T_\theta : H_{00}^{\frac{1}{2}}(\Gamma) &\longrightarrow H_{00}^{\frac{1}{2}}(\Gamma) \\ \psi &\longmapsto \psi - \theta\Big(\psi + S_2^{-1}S_1R_1\psi - U_2\Big)\end{aligned}$$

Convergence of D-N algorithm : The fixed point theorem is applied to T_θ

- The space $H_{00}^{\frac{1}{2}}(\Gamma)$ is endowed with $\langle v, w\rangle_{S_2} = \langle S_2 v, v\rangle$
- T_θ is a contraction for a suitable choice of θ

For any λ^0 on Γ, for $m > 1$ and $k = 1, 2$ solve in parallel the following problems

$$\begin{cases} -\Delta {u_k}^{m+1} & = & f & \text{in } \Omega_k \\ {u_k}^{m+1} & = & 0 & \text{on } \partial\Omega_k \backslash \Gamma \\ {u_k}^{m+1} & = & \lambda^m & \text{on } \Gamma \end{cases}$$

$$\begin{cases} -\Delta {w_k}^{m+1} & = & 0 & \text{in } \Omega_k \\ {w_k}^{m+1} & = & 0 & \text{on } \partial\Omega_k \backslash \Gamma \\ \dfrac{\partial {w_k}^{m+1}}{\partial n_k} & = & (-1)^k \left(\dfrac{\partial {u_1}^{m+1}}{\partial n_1} + \dfrac{\partial {u_2}^{m+1}}{\partial n_2} \right) & \text{on } \Gamma \end{cases}$$

$$\lambda^{m+1} = \lambda^m + \theta(w_2^{m+1} - w_1^{m+1}) \qquad \text{on } \Gamma$$

$$u_k{}^{m+1} = U_k + R_k\lambda^m$$
$$w_1^{m+1} = -S_1^{-1}\Big(S_1R_1\lambda^m + S_2\lambda^m\Big)$$
$$w_2^{m+1} = S_2^{-1}\Big(S_1R_1\lambda^m + S_2R_2\lambda^m\Big)$$

Then

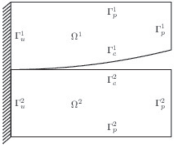
$$\lambda^{m+1} = \lambda^m + \theta(S_1^{-1} + S_2^{-1})(S_1R_1 + S_2R_2)\lambda^m$$

$$\begin{array}{rcl} T_\theta : H_{00}^{\frac{1}{2}}(\Gamma) & \longrightarrow & H_{00}^{\frac{1}{2}}(\Gamma) \\ \psi & \longmapsto & \psi + (S_1^{-1} + S_2^{-1})(S_1R_1 + S_2R_2)\psi \end{array}$$

Convergence of N-N algorithm : The fixed point theorem is applied to T_θ

- The space $H_{00}^{\frac{1}{2}}(\Gamma)$ is endowed with $\langle v, w\rangle_M = \langle Mv, v\rangle$ with $M = (S_1^{-1} + S_2^{-1})^{-1}$

T_θ is a contraction for a suitable choice of θ



- two bodies Ω^1, Ω^2 in $\mathbb{R}^2$

$$\partial\Omega^\alpha = \overline{\Gamma}_u^\alpha \cup \overline{\Gamma}_P^\alpha \cup \overline{\Gamma}_C^\alpha$$

- zero displacement on Γ_u^α
- surface traction on Γ_P^α
- contact conditions on Γ_C

- the PDEs in Ω^α, $\alpha = 1, 2$:

$$\begin{aligned} -\mathrm{div}\, \sigma^\alpha(u^\alpha) &= F^\alpha \\ \sigma^\alpha(u^\alpha) &= E^\alpha \boldsymbol{\varepsilon}(u^\alpha) \\ \boldsymbol{\varepsilon}(u^\alpha) &= \tfrac{1}{2}(\nabla u^\alpha + \nabla^\top u^\alpha) \end{aligned}$$

u^α ... displacement, σ^α ... stress
F^α ... volume force
E^α ... symmetric, elliptic elasticity tensor

- Dirichlet and Neumann b.c. :

$$\left.\begin{aligned} u^\alpha &= 0 \quad \text{on } \Gamma_u^\alpha \\ \sigma^\alpha(u^\alpha)\nu^\alpha &= P^\alpha \quad \text{on } \Gamma_P^\alpha \end{aligned}\right\} \quad \alpha = 1, 2$$

P^α ... surface traction
ν^α ... outer normal vector to $\partial\Omega^\alpha$

- unilateral contact law :

$$u_\nu \leq 0, \ \sigma_\nu \leq 0, \ \sigma_\nu u_\nu = 0$$

where $u_\nu := u^1 \cdot \nu^1 + u^2 \cdot \nu^2$ and $\sigma_\nu := \sigma^1(u^1)\nu^1 \cdot \nu^1$

- transmission of contact stresses : without friction

$$\sigma_\nu^1 = \sigma_\nu^2 = 0 \qquad \sigma_t^1 = \sigma_t^2 = 0$$

- Sets :

$$\begin{aligned}
\mathbb{V}^\alpha &= \{v^\alpha \in (H^1(\Omega^\alpha))^2 |\ v^\alpha = 0 \ \text{ on } \Gamma_u^\alpha\}, \quad \alpha = 1,2 \\
\mathbb{V} &= \mathbb{V}^1 \times \mathbb{V}^2, \quad v = (v^1, v^2) \in \mathbb{V} \\
\mathbb{K} &= \{v \in \mathbb{V} |\ v_\nu \leq 0 \ \text{ on } \Gamma_C\} \\
\mathbb{K}^2(\varphi) &= \{v^2 \in \mathbb{V}^2 |\ v^2 \cdot \nu^2 \leq -\varphi \text{ on } \Gamma_C\}, \quad \varphi \in L^2(\Gamma_C) \\
\mathbb{H}^{1/2}(\Gamma_C) &= \{\varphi \in (L^2(\Gamma_C))^2 |\ \exists v^\alpha \in \mathbb{V}^\alpha : \ \varphi = v^\alpha \ \text{ on } \Gamma_C\} \\
\mathbb{V}_0^\alpha &= \{v^\alpha \in (H^1(\Omega^\alpha))^2 |\ v^\alpha = 0 \ \text{ on } \Gamma_u^\alpha \cap \Gamma_C\}, \quad \alpha = 1,2
\end{aligned}$$

- Smoothness assumptions : $F^\alpha \in (L^2(\Omega^\alpha))^2$, $P^\alpha \in (L^2(\Gamma_p^\alpha))^2$, $\alpha = 1, 2$,

Weak formulation

$$(\mathcal{P}) \qquad \begin{cases} \textit{Find } u \in \mathbb{K} \textit{ such that} \\ a(u, v-u) \geq b(v-u) \quad \forall v \in \mathbb{K}, \end{cases}$$

where

$$a : \mathbb{V} \times \mathbb{V} \mapsto \mathbb{R}, \quad a(u,v) = a^1(u^1, v^1) + a^2(u^2, v^2),$$

$$b : \mathbb{V} \mapsto \mathbb{R}, \quad b(v) = b^1(v^1) + b^2(v^2),$$

and

$$a^\alpha(u^\alpha, v^\alpha) = \int_{\Omega^\alpha} E^\alpha \varepsilon(u^\alpha) : \varepsilon(v^\alpha)\, dx, \quad b^\alpha(v^\alpha) = \int_{\Omega^\alpha} F^\alpha \cdot v^\alpha\, dx + \int_{\Gamma_P^\alpha} P^\alpha \cdot v^\alpha\, ds, \quad \alpha = 1, 2.$$

Proposition A pair $u = (u^1, u^2) \in \mathbb{V}$ is a solution to $(\mathcal{P})$ iff u^1 and u^2 solve the following coupled problems :

$$(\mathcal{P}_1) \qquad \begin{cases} \textit{Find } u^1 \in \mathbb{K}^1(u^2 \cdot \nu^2) \textit{ such that} \\ a^1(u^1, v^1 - u^1) \geq b^1(v^1 - u^1) \quad \forall v^1 \in \mathbb{K}^1(u^2 \cdot \nu^2), \end{cases}$$

$$(\mathcal{P}_2) \qquad \begin{cases} \textit{Find } u^2 \in \mathbb{V}^2 \textit{ such that} \\ a^2(u^2, v^2) = b^2(v^2) - a^1(u^1, R^1 v^2) + b^1(R^1 v^2) \quad \forall v^2 \in \mathbb{V}^2, \end{cases}$$

where the extension mappings $R^\alpha : \mathbb{H}^{1/2}(\Gamma_C) \mapsto \mathbb{V}^\alpha$ are defined for $\lambda \in \mathbb{H}^{1/2}(\Gamma_C)$ by :

$$\begin{cases} R^\alpha \lambda \in \mathbb{V}^\alpha \quad : \quad a^\alpha(R^\alpha \lambda, v^\alpha) = 0 \quad \forall v^\alpha \in \mathbb{V}_0^\alpha \\ \qquad R^\alpha \lambda = \lambda \ \text{ on } \ \Gamma_C \end{cases}$$

Contact Neumann Algorithm

Starting for $\hat{u}_0^2$ on Γ_C. We Solve the following unilateral problem :

$$\begin{cases} -\mathrm{div}\,\sigma(u_k^1) = f^1 & \text{in } \Omega^1, \\ u_k^1 = 0 & \text{on } \Gamma_u^1, \\ \sigma(u_k^1)\cdot\nu^1 = p^1 & \text{on } \Gamma_p^1, \\ \sigma_\nu(u_k^1) \leq 0 & \text{on } \Gamma_C, \\ u_{k,\nu}^1 \leq \hat{u}_{k-1,\nu}^2 & \text{on } \Gamma_C, \\ \sigma_\nu(u_k^1)(u_{k,\nu}^1 - \hat{u}_{k-1,\nu}^2) = 0 & \text{on } \Gamma_C, \\ \sigma_t(u_k^1) = 0 & \text{on } \Gamma_C. \end{cases}$$

Contact Neumann Algorithm

Solve the Neumann problem

$$\begin{cases} -\mathrm{div}\,\sigma(u_k^2) = f^2 & \text{in } \Omega^2, \\ u_k^2 = 0 & \text{on } \Gamma_u^2, \\ \sigma(u_k^2)\cdot\nu^2 = p^2 & \text{on } \Gamma_p^2, \\ \sigma_\nu(u_k^2) = -\sigma_\nu(u_k^1) & \text{on } \Gamma_C, \\ \sigma_t(u_k^2) = 0 & \text{on } \Gamma_c. \end{cases}$$

Relaxation

$$\hat{u}_k^2 = (1-\theta)\hat{u}_{k-1}^2 + \theta u_k^2$$

If we decouple the influence of f, p and the boundary date u_0 we obtain

$$\begin{aligned}\hat{u}_2^m &= \hat{u}_2^{m-1} - \theta\Big(\hat{u}_2^{m-1} + S_2^{-1} S_1 R_1 \hat{u}_2^{m-1} - U_2\Big) \\ &= T_\theta(\hat{u}_2^{m-1})\end{aligned}$$

The operator R_1 is non-linear : mapping any function trace $v \in \mathbb{H}^{1/2}(\Gamma_C)$ to the trace $u_{|\Gamma_C}$ s.t

$$\begin{cases} -\text{div}\, \sigma(u^1) = 0 & \text{in } \Omega^1, \\ u^1 = 0 & \text{on } \Gamma_u^1, \\ \sigma(u^1)\cdot \nu^1 = p^1 & \text{on } \Gamma_p^1, \\ \sigma_\nu(u^1) \le 0 & \text{on } \Gamma_C, \\ u_\nu^1 \le v_\nu^2 & \text{on } \Gamma_C, \\ \sigma_\nu(u^1)(u_\nu^1 - v_\nu^2) = 0 & \text{on } \Gamma_C, \\ \sigma_t(u^1) = 0 & \text{on } \Gamma_C. \end{cases}$$

$$\begin{aligned} T_\theta : \mathbb{H}^{1/2}(\Gamma_C) &\longrightarrow \mathbb{H}^{1/2}(\Gamma_C) \\ \psi &\longmapsto \psi - \theta\Big(\psi + S_2^{-1} S_1 R_1 \psi - U_2\Big) \end{aligned}$$

Convergence of Contact Neumann algorithm : The fixed point theorem is applied to T_θ

- The space $\mathbb{H}^{1/2}(\Gamma_C)$ is endowed with $\langle v, w\rangle_{S_2} = \langle S_2 v, v\rangle$
- For $k = 1, 2$, S_k is bounded, bijective, self-adjoint and coercive
- The operator $S_1 R_1$ is Lipschitz and monotone

N-N Algorithm for unilateral contact problem

- consider $\lambda \in \mathbb{H}^{1/2}(\Gamma_C)$ that plays the role of the contact displacemnt
- linear elasticity for Ω^1 with the prescribed displacements λ on Γ_C

$$(\mathcal{P}_1(\lambda)) \qquad \begin{cases} \textit{Find } u^1 := u^1(\lambda) \in \mathbb{V}^1 \textit{ such that} \\ a^1(u^1, v^1) = b^1(v^1) \quad \forall v^1 \in \mathbb{V}_0^1 \\ u^1 = \lambda \quad \text{on } \Gamma_C \end{cases}$$

- Unilateral contact problem for Ω^2 with the prescribed gap λ on Γ_C

$$(\mathcal{P}_2(\lambda)) \qquad \begin{cases} \textit{Find } u^2 := u^2(\lambda) \in \mathbb{K}^2(\lambda \cdot \nu^1) \textit{ such that} \\ a^2(u^2, v^2 - u^2) \geq b^2(v^2 - u^2) \quad \forall v^2 \in \mathbb{K}^2(\lambda \cdot \nu^1) \end{cases}$$

- Neumann problem on Γ_C for Ω^1

$$(\mathcal{P}_3(\lambda)) \quad \left\{ \begin{array}{l} \textit{Find } w^1 := w^1(\lambda) \in \mathbb{V}^1 \textit{ such that} \\ a^1(w^1, v^1) = \frac{1}{2}(-a^1(u^1, v^1) + b^1(v^1) - a^2(u^2(\lambda), \pi^2 v^1) + b^2(\pi^2 v^1)) \\ \hfill \forall v^1 \in \mathbb{V}^1 \end{array} \right.$$

- Neumann problem on Γ_C for Ω^2

$$(\mathcal{P}_4(\lambda)) \quad \left\{ \begin{array}{l} \textit{Find } w^2 := w^2(\lambda) \in \mathbb{V}^2 \textit{ such that} \\ a^2(w^2, v^2) = \frac{1}{2}(a^2(u^2, v^2) - b^2(v^2) + a^1(u^1(\lambda), \pi^1 v^2) - b^1(\pi^1 v^2)) \\ \hfill \forall v^2 \in \mathbb{V}^2 \end{array} \right.$$

Algorithm (DD)

Let $\lambda_0 \in \mathbb{H}^{1/2}(\Gamma_C)$ and $\theta > 0$ be given.

For $k \geq 1$ integer, define u_k^α, w_k^α, $\alpha = 1, 2$ and λ_k by :

<u>*Step 1.*</u> $u_k^1 \in \mathbb{V}^1$ solves $(\mathcal{P}_1(\lambda_{k-1}))$

<u>*Step 2.*</u> $u_k^2 \in \mathbb{K}^2(\lambda_{k-1} \cdot \nu^1)$ solves $(\mathcal{P}_2(\lambda_{k-1}))$

<u>*Step 3.*</u> $w_k^1 \in \mathbb{V}^1$ solves $(\mathcal{P}_3(\lambda_{k-1}))$

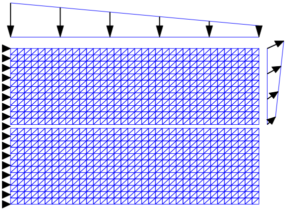
<u>*Step 4.*</u> $w_k^2 \in \mathbb{V}^2$ solves $(\mathcal{P}_4(\lambda_{k-1}))$

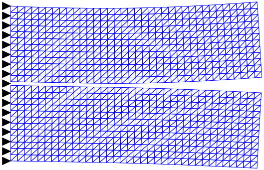
<u>*Step 5.*</u> $\lambda_k = \lambda_{k-1} + \theta(w_k^1 - w_k^2)$ on Γ_C

- choice of the update parameter θ crucial for the convergence

Numerical Experiments

- <u>Geometry :</u> for n, m DOFs with $\theta = 0.2$





- outer iterations : out
- inner iterations : inn
- CPU time [sec.] : $time$
- final precision of the contact displacement : ε_λ

Tab.: N-N Algorithm fot Contact problem for $\theta = 0.4$.

n	m	*time*	*out/inn*	ε_λ
24	6	0.01	15/123	6.4e−6
72	12	0.02	15/135	6.6e−6
240	24	0.03	15/148	4.1e−6
864	48	0.11	17/167	8.0e−6
3264	96	0.67	16/168	7.9e−6
12672	192	6.41	18/193	2.5e−6
49920	384	44.41	16/181	7.7e−6

TAB.: N-N ALGORITHM FOT CONTACT PROBLEM, *out/inn* for various θ.

n	m	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$	$\theta = 0.5$	$\theta = 0.6$	$\theta = 0.7$
24	6	46/307	23/166	14/114	15/123	19/156	18/145	37/290
72	12	47/336	32/241	22/178	15/135	17/155	21/187	39/326
240	24	45/323	31/249	23/199	15/148	19/179	23/205	51/461
864	48	50/381	29/247	23/208	17/167	18/176	31/279	57/507
3264	96	49/383	29/251	24/231	16/168	18/187	24/235	51/486
12672	192	49/391	30/276	23/231	18/193	18/186	25/257	63/616
49920	384	48/388	30/283	25/257	16/181	17/181	25/257	61/768

References

- C. Eck, B. Wohlmuth. *Convergence of a Contact-Neumann iteration for the solution of two-body contact problems.* Mathematical Models and Methods in Applied Sciences, 13 (2003).
- G. Bayada, J. Sabil, T. Sassi. *A Neumann-Neumann domain decomposition algorithm for the Signorini problem.* Appl. Math. Letters, 17 (2004).
- J. Haslinger, R. Kucera, T. Sassi, *A domain Decomposition Algorithm for contact problems : Analysis and Implementation*, Math. Model. Nat. Phenom., 2008