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Lecture 1 : Introduction to Domain Decomposition Methods
Lecture 2 : Neumann-Neumann DD Algorithm
Lecture 3 : DDM for Contact Problems

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Outline

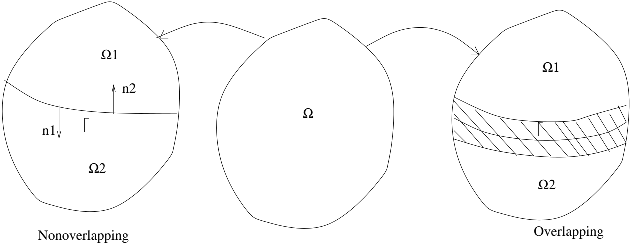
1. Motivations
2. The model problem
3. Schwarz Domain Decomposition
4. Nonoverlapping Domain Decomposition
5. Basic Domain Decomposition Algorithms

Motivations

- Simplification of problems on complicated geometry
- The separation of the physical domain into regions that can be modeled with different equations : very convenient framework for the solution of heterogeneous and multiphysics problems gouverned by PDEs
- Combine DDM and any disretization method for PDEs (FEM, FV for example) to make their algebraic solution more efficient on parallel comptuer.

The idea

1. The physical domain Ω is divided into two or more subdomains on which discretized problems of smaller dimension are to be solved
2. Parallel algorithms can be used to solve the resulting subproblems
3. There are two ways to divide Ω :
    - with nonoverlapping subdomains
    - with overlapping subdomains


Ω1
n2
n1
Γ
Ω2
Nonoverlapping
Ω
Ω1
Γ
Ω2
Overlapping

Consider the Poisson equation on a region $\Omega \in \mathbb{R}^2$. Let $f \in L^2(\Omega)$ be given. Find $u : \Omega \longrightarrow \mathbb{R}$ s.t.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

The weak formulation reads : find $u \in V$ s.t.

$$a(u,v) = (f,v) \qquad \forall v \in V$$

where

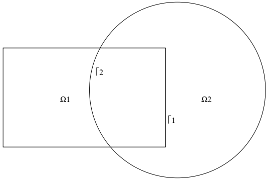
$$V = H_0^1(\Omega) = \{v \in H^1(\Omega), \quad v = 0 \text{ on } \partial\Omega\}$$

$$a(u,v) = \int_\Omega \nabla u \nabla v \, dx \qquad\qquad (f,v) = \int_\Omega fv \, dx$$

$a(\cdot,\cdot)$ is the bilinear form associated to the Laplace operator.

Schwarz Domain Decomposition

The first Domain Decomposition Method was introduced by Schwarz in 1870 for a complicated domain, composed for two simple ones, a disk and a rectangle.



To show that the equation

$$\Delta u = 0 \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega$$

can be integrated with arbitrary boundary conditions. Schwarz proposed an iterative method which uses solutions in a disk and rectangle

- Start with initial guess u_1^0 along Γ_1, for $m \geq 1$ compute iteratively $u_1^{m+1}$and u_2^{m+1} according to the algorithm (for simplicity, we omit the Dirichlet BC on $\partial\Omega$

$$\begin{cases} -\Delta u_1^{m+1} = 0 & \text{in } \Omega_1 \\ u_1^{m+1} = u_2^m & \text{on } \Gamma_1 \end{cases}$$

$$\begin{cases} -\Delta u_2^{m+1} = 0 & \text{in } \Omega_1 \\ u_2^{m+1} = u_1^{m+1} & \text{on } \Gamma_2 \end{cases}$$

- DD 22 : International Conference on Domain Decomposition has been held in 13 countries (DD 1, Paris 1987).

Suppose that Ω is partitioned into two nonoverlapping subdomains Ω_1 and Ω_2

$$\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2} \qquad \Omega_1 \cap \Omega_2 = \emptyset \qquad \Gamma = \partial\Omega_1 \cap \partial\Omega_2$$

Problem (1) is equivalent to the following coupled problem :

$$\left\{ \begin{array}{lcll} -\Delta u_1 & = & f & \text{in } \Omega_1 \\ u_1 & = & 0 & \text{on } \partial\Omega_1 \backslash \Gamma \end{array} \right.$$

$$\begin{array}{lcll} u_1 & = & u_2 & \text{on } \Gamma \\ \dfrac{\partial u_1}{\partial n_1} & = & -\dfrac{\partial u_2}{\partial n_2} & \text{on } \Gamma \end{array}$$

$$\left\{ \begin{array}{lcll} -\Delta u_2 & = & f & \text{in } \Omega_2 \\ u_2 & = & 0 & \text{on } \partial\Omega_2 \backslash \Gamma \end{array} \right.$$

- u_i is the restriction of u on Ω_i
- n_i the outward normal to Ω_i
- the conditions on the interfce Γ are called *transmission conditions* :
  - the continuity of the solution at the interface Γ
  - the continuity of the normal derivative (the flux) at the interface Γ

Theorem

The function u is the solution of (1) if and only if the functions $u_1 \in V_1$ and $u_2 \in V_2$ satisfy the following split problem

$$a_1(u_1, v) = (f, v)_1 \qquad \forall v \in V_1^0 \tag{2}$$

$$u_1 = u_2 \qquad \text{on } \Gamma \tag{3}$$

$$a_2(u_2, v) = (f, v)_2 - a_1(u_1, R_1\gamma(v)) + (f, R_1\gamma(v))_1 \; \forall v \in V_2 \tag{4}$$

- $(f, v)_k$ the scalar product of $L^2(\Omega_k)$ $(k = 1, 2)$
- $\gamma(v)$ the trace on Γ of fuction $v \in H^1(\Omega)$
- $V_k = \{v \in H^1(\Omega_k), v = 0 \text{ on } \partial\Omega \cap \partial\Omega_k\}, \quad V_0^k = H_0^1(\Omega_k)$
- R_k is the harmonic extention operator of the trace to Ω_k

For $\varphi \in \Gamma$, $R_k \in \Omega_k$ s.t.

$$\begin{aligned} a_k(R_k\varphi, v) &= 0 \qquad \forall v \in V_k^0 \\ R_k\varphi &= \varphi \qquad \text{on } \Gamma \end{aligned}$$

Proof

$\Rightarrow$) If u is the solution of (1) then $u_1 \in V_1$ and $u_2 \in V_2$

- Condition $u_1 = u_2$ is automatically verified since $u \in H^1(\Omega)$
- For any $v \in V_1^0$, choose $\tilde{v} = (v, 0) \in H_0^1(\Omega)$ as a test function in (1), then (2) is satisfied.
- For $v \in V_2$, let $\tilde{v} = (R_1\gamma(v), v) \in H_0^1(\Omega)$ s.t. $\tilde{v} = v \in \Omega_2$ and $\tilde{v} = R_1\gamma(v) \in \Omega_1$, then (4) is satisfied.

$\Leftarrow$ Viceversa

- Clearly, $u \in H_0^1(\Omega)$.
- For $v \in H_0^1(\Omega)$, take $v_1 = v - R_1\gamma(v)$ in Ω_1 and $v_2 = v$ in Ω_2, note that $v_1 \in V_1^0$, $v_2 \in V_2$, then

$$\begin{aligned} a(u,v) &= a_1(u_1, v_1 + R_1\gamma(v)) + a_2(u_2, v_2) \\ &= a_1(u_1, v_1) + a_1(u_1, R_1\gamma(v)) + a_2(u_2, v_2) \\ &= (f, v_1)_1 + (f, v_2)_2 + (f, R_1\gamma(v))_1 \\ &= (f, v_1 + R_1\gamma(v))_1 + (f, v_2)_2 \\ &= (f, v) \end{aligned}$$

Remark

simple integration by parts in (4) shows

$$\int_\Gamma \frac{\partial u_2}{\partial n_2} v d\Gamma = -\int_\Gamma \frac{\partial u_1}{\partial n_2} v d\Gamma \qquad \forall v \in V_2$$

Problem (2) is the variational formulation of

$$\begin{cases} -\Delta u_1 = f & \text{in } \Omega_1 \\ u_1 = 0 & \text{on } \partial\Omega_1 \backslash \Gamma \\ u_1 = u_2 & \text{on } \Gamma \end{cases}$$

Problem (4) is the variational formulation of

$$\begin{cases} -\Delta u_2 = f & \text{in } \Omega_2 \\ u_2 = 0 & \text{on } \partial\Omega_2 \backslash \Gamma \\ \dfrac{\partial u_2}{\partial n_2} = -\dfrac{\partial u_1}{\partial n_1} & \text{on } \Gamma \end{cases}$$

Given u_2^0 on Γ, for $m > 1$ solve the problems

$$\begin{cases} -\Delta {u_1}^m = f & \text{in } \Omega_1 \\ {u_1}^m = 0 & \text{on } \partial\Omega_1 \backslash \Gamma \\ {u_1}^m = {u_2}^{m-1} & \text{on } \Gamma \end{cases}$$

$$\begin{cases} -\Delta {u_2}^m = f & \text{in } \Omega_2 \\ {u_2}^m = 0 & \text{on } \partial\Omega_2 \backslash \Gamma \\ \dfrac{\partial {u_2}^m}{\partial n_2} = \dfrac{\partial {u_1}^m}{\partial n_1} & \text{on } \Gamma \end{cases}$$

The equivalence theorem guarantees that when the two sequences (u_1^m) and (u_2^m) converge, then their limit will be necessarily the solution to the exact problem. The DN algorithm is therefore consistent.

A variant of the DN algorithm can be set up by replacing the Dirichlet condition in the first subdomain by

$$u_1^m = \theta u_2^{m-1} + (1-\theta) u_1^{m-1} \qquad \text{on } \Gamma$$

that is by introducing a relaxation which depends on a positive parameter θ. In such a way it is always possible to reduce the error between two subsequent iterates.
In general, there exists a suitable value $\theta_{max} < 1$ such that the DN algorithm converges for any possible choice of the relaxation parameter θ in the interval $(0, \theta_{max})$.

Ω is divided into two nonoverlapping subdomains. Denote by λ the (unknown) value of the solution u at their interface Γ. For any λ^0 on Γ, for $m > 1$ and $k = 1, 2$ solve in parallel the following problems

$$\begin{cases} -\Delta {u_k}^{m+1} = f & \text{in } \Omega_k \\ {u_k}^{m+1} = 0 & \text{on } \partial\Omega_k \backslash \Gamma \\ {u_k}^{m+1} = \lambda^m & \text{on } \Gamma \end{cases}$$

$$\begin{cases} -\Delta {w_k}^{m+1} = f & \text{in } \Omega_k \\ {w_k}^{m+1} = 0 & \text{on } \partial\Omega_k \backslash \Gamma \\ \dfrac{\partial {w_k}^{m+1}}{\partial n_k} = (-1)^k \left(\dfrac{\partial {u_1}^{m+1}}{\partial n_1} + \dfrac{\partial {u_2}^{m+1}}{\partial n_2} \right) & \text{on } \Gamma \end{cases}$$

$$\lambda^{k+1} = \lambda^k - \theta(w_1^{m+1} - w_2^{m+1}) \qquad \text{on } \Gamma$$

where θ is a positive acceleration parameter.

For $m > 0$ solve in parallel the following problems

$$\begin{cases} -\Delta u_1^{m+1} & = & f & \text{in } \Omega_1 \\ u_1^{m+1} & = & 0 & \text{on } \partial\Omega_1 \backslash \Gamma \\ \dfrac{\partial {u_1}^{m+1}}{\partial n_1} + \alpha_1 {u_1}^{m+1} & = & \dfrac{\partial {u_2}^{m}}{\partial n_2} + \alpha_2 {u_2}^{m} & \text{on } \Gamma \end{cases}$$

$$\begin{cases} -\Delta u_2^{m+1} & = & f & \text{in } \Omega_2 \\ u_2^{m+1} & = & 0 & \text{on } \partial\Omega_2 \backslash \Gamma \\ \dfrac{\partial {u_2}^{m+1}}{\partial n_2} + \alpha_2 {u_2}^{m+1} & = & \dfrac{\partial {u_1}^{m}}{\partial n_1} + \alpha_1 {u_1}^{m} & \text{on } \Gamma \end{cases}$$

where α_1 and α_2 are non-negative acceleration parameters.

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