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Shape derivatives without taking the shape derivative

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$$\min_{\Omega} J(\Omega) = \frac{1}{2}\int_{\Omega} |\nabla u_D - \nabla u_N|^2$$

where

$$\begin{aligned} -\Delta u_D &= 0, & -\Delta u_N &= 0, & &\text{in } \Omega \\ u_D &= 1, & u_N &= 1, & &\text{on } \Gamma_f, \\ u_D &= 0, & \frac{\partial u_N}{\partial \nu} &= \lambda & &\text{on } \Gamma, \end{aligned}$$

Find u_D, $u_N \in H^1(\Omega)$ s. th. $u_D = u_N = 1$ on Γ_f, $u_D = 0$ on Γ

$$\begin{aligned} (\nabla u_D, \nabla v) &= 0, & v &\in H_0^1(\Omega), \\ (\nabla u_N, \nabla v) &= \int_{\Gamma} \lambda v, & v &\in V = \{v \in H^1(\Omega) \colon v = 0 \text{ on } \Gamma_f\}. \end{aligned}$$

by elliptic regularity: u_D, $u_N \in H^2(\Omega)$

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Recall: u_N, u_N^t satisfy

$$(\nabla u_N, \nabla v) = \lambda \int_\Gamma v$$
$$(A_t \nabla u_N^t, \nabla v) = \lambda \int_\Gamma \omega_t v, \qquad v \in V.$$

uniform boundedness of u_n^t:

$$(A_t \nabla (u_N^t - u_N), \nabla v) = \lambda \int_\Gamma \omega_t v - (A_t \nabla u_N, \nabla v), \quad v \in V.$$

Since $v = u_N^t - u_N \in V$ and A_t is uniformly coercive

$$\frac{1}{2} \|\nabla (u_N^t - u_N)\|^2 \leq |\lambda| \int_\Gamma \omega_t |u_N^t - u_N| + |\big(A_t \nabla u_N, \nabla (u_N^t - u_N)\big)|$$

a bound for $u_N^t - u_N$ follows.

Continuity: $\lim_{t\to 0} u_N^t = u_N$ Use the equations for u_N^t and u_N

$$(\nabla(u_N^t - u_N), \nabla v) = (A_t \nabla u_N^t, \nabla v) - \big((A_t - I)\nabla u_N^t, \nabla v\big) - (\nabla u_N, \nabla v)$$
$$= -(A_t - I)\nabla u_N^t), \nabla v) + \lambda \int_\Gamma (\omega_t - 1) v$$

Choose $v = u_N^t - u_N$

$$\|\nabla(u_N^t - u_N)\| \le \|A_t - I\|_{L^\infty} \|\nabla u_N^t\| + |\lambda| \, \|\omega_t - 1\|_{L^2} \, \|\tau_\Gamma\|$$

Material derivative of u_N^t: Choose a sequence $t_k \to 0$ and set

$$y_k = \frac{1}{t_k}(u_N^{t_k} - u_N).$$

Then

$$(\nabla y_k, \nabla v) = -(\frac{1}{t_k}(A_{t_k} - I)\nabla u_N^{t_k}, \nabla v) + \lambda \int_\Gamma \frac{1}{t_k}(\omega_{t_k} - 1)v \qquad (*)$$

As before (y_k) is bounded in V

$\Rightarrow$ subsequence (y_k) and $y \in V$ such that

$$y_k \rightharpoonup y$$

Passing to the limit y satisfies

$$(\nabla y, \nabla v) = -(A\nabla u_N, \nabla v) + \lambda \int_\Gamma v \operatorname{div}_\Gamma h$$

By uniqueness y is independent of (t_k)

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Choose $v = y_k$ in $(*)$

$$\|\nabla y_k\|^2 = -\Big(\frac{1}{t_k}(A_{t_k} - I)\nabla u_N^{t_k}, \nabla y_k\Big) + \lambda \int_\Gamma \frac{1}{t_k}(\omega_{t_k} - 1)y_k$$

trace operator $\tau_\Gamma \in \mathcal{L}(V, H^{1/2}(\Gamma))$ hence

$$\tau_\Gamma y_k \rightharpoonup \tau_\Gamma y \qquad \text{in } H^{1/2}(\Gamma),$$

thus by compact embedding strongly in $L^2(\Gamma)$.

Passing to the limit

$$\lim_{k\to\infty} \|\nabla y_k\|^2 = -(A\nabla u_N, \nabla y) + \lambda \int_\Gamma y \operatorname{div}_\Gamma h = \|y\|^2$$

This implies $y_k \to y$ strongly in V.

Using the shape derivative:

Apply

$$\frac{d}{dt}\int_{\Omega_t} f(t,x)|_{t=0} = \int_{\Omega} f_t(0,x) + \int_{\Gamma} f(0,x)\, h\cdot\nu$$

to

$$J(\Omega_t) = \frac{1}{2}\int_{\Omega_t} |\nabla u_{D,t} - \nabla u_{N,t}|^2$$

gives

$$\begin{aligned} dJ(\Omega)h &= \int_{\Omega} \nabla(u'_D - u'_N)\cdot\nabla(u_D - u_N) + \frac{1}{2}\int_{\Gamma} |\nabla(u_D - u_N)|^2\, h\cdot\nu \\ &= \int_{\Omega} \nabla u'_D\cdot\nabla(u_D - u_N) - \int_{\Omega} \nabla u'_N\cdot\nabla(u_D - u_N) \\ &+ \frac{1}{2}\int_{\Gamma} |\nabla(u_D - u_N)|^2\, h\cdot\nu \end{aligned}$$

where

$$u'_D = \dot{u}_D - \nabla u_D\cdot h, \qquad u'_N = \dot{u}_N - \nabla u_N\cdot h.$$

One can show

$$
\begin{aligned}
-\Delta u_D' &= 0, & -\Delta u_N' &= 0, & &\text{in } \Omega\\
u_D' &= 0, & u_N' &= 0, & &\text{on } \Gamma_f,\\
u_D' &= -\frac{\partial u_D}{\partial \nu} h\cdot\nu, & \frac{\partial u_N'}{\partial \nu} &= \mathrm{div}_\Gamma (h\cdot\nu \nabla_\Gamma u_N) + \lambda\kappa\, h\cdot\nu & &\text{on } \Gamma,
\end{aligned}
$$

Apply Green's theorem

$$
\begin{aligned}
dJ(\Omega)h &= \int_\Gamma u_D' \frac{\partial}{\partial \nu}(u_D - u_N) - \int_\Gamma \frac{\partial u_N'}{\partial \nu}(u_D - u_N)\\
&\quad + \frac{1}{2}\int_\Gamma |\nabla (u_D - u_N)|^2\, h\cdot\nu\\
&= -\int_\Gamma ((\frac{\partial u_D}{\partial \nu})^2 - \lambda \frac{\partial u_D}{\partial \nu}) h\cdot\nu + \int_\Gamma (\mathrm{div}_\Gamma (h\cdot\nu\nabla_\Gamma u_N) + \lambda\kappa\, h\cdot\nu) u_N\\
&\quad + \frac{1}{2}\int_\Gamma |\nabla (u_D - u_N)|^2\, h\cdot\nu.
\end{aligned}
$$

Since

$$\int_\Gamma \mathrm{div}_\Gamma(h\cdot\nu\nabla_\Gamma u_N)u_N = -\int h\cdot\nu|\nabla_\Gamma u_N|^2 = -\int_\Gamma (\frac{\partial u_N}{\partial\tau})^2\, h\cdot\nu$$

and

$$\frac{1}{2}\int_\Gamma |\nabla(u_D-u_N)|^2\, h\cdot\nu = \frac{1}{2}\int_\Gamma (|\nabla u_D|^2+|\nabla u_N|^2-\nabla u_D\cdot\nabla u_N)$$

$$= \frac{1}{2}\int_\Gamma ((\frac{\partial u_D}{\partial\nu})^2+\lambda^2+(\frac{\partial u_N}{\partial\tau})^2-\lambda\frac{\partial u_D}{\partial\nu})$$

we eventually obtain

$$dJ(\Omega)h = \frac{1}{2}\int_\Gamma (\lambda^2-(\frac{\partial u_D}{\partial\nu})^2-(\frac{\partial u_N}{\partial\tau})^2+2\lambda\kappa u_N)\, h\cdot\nu.$$

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Recall $DJ(\Omega)h = \lim_{t\to 0} \frac{1}{t}(J(\Omega_t) - J(\Omega))$
where

$$J(\Omega_t) = \frac{1}{2}\int_{\Omega_t} |\nabla(u_{D,t} - u_{N,t})|^2$$

Set $z_t = u_{D,t} - u_{N,t}$, $z = u_D - u_N$, then

$$J(\Omega_t) = \frac{1}{2}\int_{\Omega_t} |\nabla z_t|^2 = \frac{1}{2}\int_{\Omega} A_t \nabla z^t \cdot \nabla z^t$$

Hence

$$\begin{aligned} J(\Omega_t) - J(\Omega) &= \frac{1}{2}\int_{\Omega}(A_t\nabla z^t \cdot \nabla z^t - |\nabla z|^2) \\ &= \frac{1}{2}\int_{\Omega}(A_t - I)\nabla z^t \cdot \nabla z^t + \frac{1}{2}\int_{\Omega}(|\nabla z^t|^2 - |\nabla z|^2) \\ &= J_1(t) + J_2(t) \end{aligned}$$

Since $J_1(0) = 0$

$$\dot{J}_1(0) = \lim_{t\to 0} \frac{1}{2}\int_\Omega \frac{1}{t}(A_t - I)\nabla z^t \cdot \nabla z^t = \frac{1}{2}\int_\Omega A\nabla z \cdot \nabla z$$
$$= \frac{1}{2}\int_\Omega (A\nabla u_D \cdot \nabla u_D + A\nabla u_N \cdot \nabla u_N) - \int_\Omega A\nabla u_D \cdot \nabla u_N$$

where $A = I\,\mathrm{div}\,h - (\nabla h)^T - \nabla h$.

For $J_2(t)$ use $a^2 - b^2 = (a-b)^2 + 2b(a-b)$

$$J_2(t) = \frac{1}{2}\int_\Omega (|\nabla z^t|^2 - |\nabla z|^2)$$
$$= \frac{1}{2}\int_\Omega |\nabla(z^t - z)|^2 + \int_\Omega \nabla z \cdot \nabla(z^t - z)$$
$$= J_{21}(t) + J_{22}(t)$$

Then ($t > 0$)

$$\frac{1}{t} J_{21}(t) = \frac{1}{2} \int_\Omega \frac{1}{t} |\nabla (z^t - z)|^2$$
$$\leq \int_\Omega \frac{1}{t} |\nabla (u_D^t - u_D|^2 + \int_\Omega \frac{1}{t} |\nabla (u_N^t - u_N|^2$$

Hence

$$\dot{J}_{21}(0) = 0$$

provided

$$\lim_{t \to 0} \frac{1}{t} \|u_D^t - u_D\|_{H^1}^2 = \lim_{t \to 0} \frac{1}{t} \|u_N^t - u_N\|_{H^1}^2 = 0,$$

i.e. if states depend Hölder continuously on t.

Consider

$$J_{22}(t) = \frac{1}{2}\int_\Omega \nabla z \cdot \nabla(z^t - z)$$
$$= \int_\Omega \nabla(u_D - u_N) \cdot \nabla\big((u_D^t - u_D) - (u_N^t - u_N)\big)$$
$$= \int_\Omega \nabla u_D \cdot \nabla(u_D^t - u_D) - \int_\Omega \nabla u_N \cdot \nabla(u_D^t - u_D)$$
$$- \int_\Omega \nabla(u_D - u_N) \cdot \nabla(u_N^t - u_N).$$

Since $u_D^t - u_D \in H_0^1(\Omega)$

$$\int_\Omega \nabla u_D \cdot \nabla(u_D^t - u_D) = 0$$
$$\int_\Omega \nabla u_N \cdot \nabla(u_D^t - u_D) = \lambda \int_\Gamma (u_D^t - u_D) = 0.$$

Hence

$$J_{22}(t) = \int_\Omega \nabla u_N \cdot \nabla(u_D - u_N) - \int_\Omega \nabla u_N^t \cdot \nabla(u_D - u_N).$$

Recall $u_D - u_N \in V$, therefore

$$\begin{aligned} J_{22}(t) = &\int_\Omega \nabla u_N \cdot \nabla(u_D - u_N) - \int_\Omega A_t \nabla u_N^t \cdot \nabla(u_D - u_N) \\ &+ \int_\Omega (A_t - I)\nabla u_N^t \cdot \nabla(u_D - u_N) \\ &= -\lambda \int_\Gamma (w_t - 1)(u_D - u_N) + \int_\Omega (A_t - I)\nabla u_N^t \cdot \nabla(u_D - u_N) \end{aligned}$$

Hence

$$\dot{J}_{22}(0) = \lambda \int_\Gamma u_N \operatorname{div}_\Gamma h + \int_\Omega A \nabla u_N \cdot \nabla(u_D - u_N).$$

Collecting terms

$$
\begin{aligned}
dJ(\Omega)h &= \dot{J}_1(0) + \dot{J}_{21}(0) + \dot{J}_{22}(0) \\
&= \frac{1}{2}\int_\Omega (A\nabla u_D \cdot \nabla u_D + A\nabla u_N \cdot \nabla u_N) - \int_\Omega A\nabla u_D \cdot \nabla u_N \\
&+ \lambda \int_\Gamma u_N \,\mathrm{div}_\Gamma\, h + \int_\Omega A\nabla u_N \cdot \nabla(u_D - u_N) \\
&= \frac{1}{2}\int_\Omega (A\nabla u_D \cdot \nabla u_D - A\nabla u_N \cdot \nabla u_N) + \lambda \int_\Gamma u_N \,\mathrm{div}_\Gamma\, h
\end{aligned}
$$

which is identical to the term obtained using material derivatives. As before

$$
DJ(\Omega)h = \frac{1}{2}\int_\Gamma (\lambda^2 - (\frac{\partial u_D}{\partial \nu})^2 + 2\lambda\kappa u_N - (\frac{\partial u_N}{\partial \nu})^2)\, h \cdot \nu
$$

Shape optimization approach

Neumann formulation

$$\min_{\Gamma} J(\Gamma) \equiv \frac{1}{2} \int_{\Gamma} u_N^2 \, d\sigma$$

subject to:

$$-\Delta u_N = 0 \qquad \text{in } \Omega,$$

$$u_N = 1 \qquad \text{on } \Gamma_f$$

$$\frac{\partial u_N}{\partial \nu} = \lambda \qquad \text{on } \Gamma$$

Henceforth: $u_N \rightsquigarrow u$

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$DJ(\Gamma)h = \lim_{t\to 0} \frac{1}{t}(J(\Gamma_t) - J(\Gamma))$

where

$$J(\Gamma_t) = \int_{\Gamma_t} u_t^2 = \int_\Gamma \omega_t (u^t)^2.$$

Hence

$$\begin{aligned} J(\Gamma_t) - J(\Gamma) &= \frac{1}{2}\int_\Gamma (\omega_t (u^t)^2 - u^2) \\ &= \frac{1}{2}\int_\Gamma (\omega_t - 1)(u^t)^2 + \frac{1}{2}\int_\Gamma ((u^t)^2 - u^2) \\ &= J_1(t) + J_2(t) \end{aligned}$$

and

$$\dot{J}_1(0) = \frac{1}{2}\int_\Gamma u^2 \operatorname{div}_\Gamma h.$$

Use $a^2 - b^2 = (a-b)^2 + 2b(a-b)$ again in

$$\begin{aligned} J_2(t) &= \frac{1}{2}\int_\Gamma ((u^t)^2 - u^2) \\ &= \frac{1}{2}\int_\Gamma (u^t - u)^2 + \int_\Gamma u(u^t - u) \\ &= J_{21}(t) + J_{22}(t). \end{aligned}$$

By the trace theorem

$$|\frac{1}{t}J_{21}(t)| = |\frac{1}{2}\int_\Gamma \frac{1}{t}(u^t - u)^2| \leq \frac{1}{2}|\tau_\Gamma|\,\frac{1}{|t|}\|u^t - u\|^2_{H^1}$$

By Hölder continuity of $t \to u^t$

$$\dot{J}_{21}(0) = 0.$$

However

$$J_{22}(t) = \int_\Gamma u(u^t - u)$$

would lead to a term involving $\dot{u}$. Therefore, introducing the adjoint state p

$$\int_\Omega \nabla p \cdot \nabla v = \int_\Gamma uv, \qquad v \in V$$

i.e.

$$\begin{aligned} -\Delta p &= 0, \quad \text{in } \Omega \\ p &= 0, \quad \text{on } \Gamma_f, \\ \frac{\partial p}{\partial \nu} &= u, \quad \text{on } \Gamma \end{aligned}$$

Green's theorem leads to

$$\begin{aligned} J_{22}(t) &= \int_\Gamma u(u^t - u) = \int_{\partial\Omega} \frac{\partial p}{\partial \nu}(u^t - u) \\ &= \int_\Omega \nabla(u^t - u)\cdot\nabla p = \int_\Omega \nabla u^t \cdot \nabla p - \int_\Omega \nabla u \cdot \nabla p \\ &= \int_\Omega A_t \nabla u^t \cdot \nabla p - \int_\Omega \nabla u \cdot \nabla p - \int_\Omega (A_t - I)\nabla u^t \cdot \nabla p \\ &= \lambda \int_\Gamma (\omega_t - 1) p - \int_\Omega (A_t - I)\nabla u^t \cdot \nabla p. \end{aligned}$$

Therefore

$$\dot{J}_{22}(0) = \lambda \int_\Gamma p \, \mathrm{div}_\Gamma \, h - \int_\Omega A \nabla u \cdot \nabla p.$$

Recall

$$\int_\Omega A\nabla u\cdot\nabla p = \int_\Omega \big(\Delta u\,(h\cdot\nabla p) + \Delta p\,(h\cdot\nabla u)\big)$$
$$- \int_\Gamma \big(\frac{\partial u}{\partial\nu}\,(h\cdot\nabla p) + \frac{\partial p}{\partial \nu}\,(h\cdot\nabla u)\big) + \int_\Gamma \nabla u\cdot\nabla p\,(h\cdot\nu)$$
$$= -\int_\Gamma (h\cdot\lambda\nabla p + h\cdot u\nabla u) + \int_\Gamma \nabla u\cdot\nabla p\, h\cdot\nu$$

Hence

$$\dot{J}_{22}(0) = \int_\Gamma \lambda p\,\mathrm{div}_\Gamma\, h + \int_\Gamma (h\cdot\lambda\nabla p + h\cdot u\nabla u) - \int_\Gamma \nabla u\cdot\nabla p\, h\cdot\nu$$
$$= \int_\Gamma \lambda p\,\mathrm{div}_\Gamma\, h + \int_\Gamma h\cdot\nabla(\lambda p + \frac{1}{2}u^2) - \int_\Gamma \nabla u\cdot\nabla p\, h\cdot\nu$$

Collecting terms

$$
\begin{aligned}
DJ(\Gamma)h &= \dot{J}_1(0) + \dot{J}_2(0) \\
&= \int_\Gamma (\mathrm{div}_\Gamma\, h\,(\frac{1}{2}u^2 + \lambda p) + h \cdot \nabla(\frac{1}{2}u^2 + \lambda p) - \int_\Gamma \nabla u \cdot \nabla p\, h \cdot \nu \\
&= \int_\Gamma (\mathrm{div}_\Gamma\, h\,(\frac{1}{2}u^2 + \lambda p) + h \cdot \nabla_\Gamma(\frac{1}{2}u^2 + \lambda p) \\
&\quad + \frac{\partial}{\partial \nu}(\frac{1}{2}u^2 + \lambda p) - \int_\Gamma \nabla u \cdot \nabla p\, h \cdot \nu.
\end{aligned}
$$

Tangential Green's fromula

$$
\int_\Gamma (\varphi\, \mathrm{div}_\Gamma\, h + h \cdot \nabla_\Gamma \varphi) = \int_\Gamma \kappa\varphi\, h \cdot \nu
$$

and

$$
\frac{\partial}{\partial \nu}(\frac{1}{2}u^2 + \lambda p) = u\frac{\partial u}{\partial \nu} + \lambda \frac{\partial p}{\partial \nu} = 2\lambda u
$$

lead to

$$DJ(\Gamma)h = \int_\Gamma (\kappa(\frac{1}{2}u^2 + \lambda p) - \nabla u \cdot \nabla p + 2\lambda u)\, h \cdot \nu.$$

where

$$\begin{aligned} -\Delta u &= 0, & -\Delta p &= 0, & &\text{in } \Omega \\ u &= 1, & p &= 0, & &\text{on } \Gamma_f, \\ \frac{\partial u}{\partial \nu} &= \lambda, & \frac{\partial p}{\partial \nu} &= u & &\text{on } \Gamma, \end{aligned}$$

$$\min J(u,\Omega) = \int_\Omega j_1(u)\,dx + \int_\Gamma j_2(u)\,d\sigma$$

subject to:

$$E(u,\Omega) = 0.$$

Embed into family of perturbed problems on Ω_t defined by

$$T_t = id + th$$

$$\Omega_t = T_t(\Omega), \qquad \Gamma_t = T_t(\Gamma)$$

and

$$E(u_t,\Omega_t) = 0,$$

referred to the reference domain by method of mapping

$$\tilde{E}(u^t,t) = 0.$$

Hence $u_0 = u = u^0$ and

$$E(u_t, \Omega_t) = 0 \Leftrightarrow \tilde{E}(u^t, t) = 0.$$

Example:

$$E(u_t, \Omega_t) = 0 \quad \cdots - \Delta u_t = f_t \qquad \text{on } \Omega_t$$

$$\tilde{E}(u^t, t) = 0 \quad \cdots - \operatorname{div}(A_t \nabla u^t)) = f^t \qquad \text{on } \Omega$$

(H1) $\tilde{E}\colon X \times (-\tau, \tau) \to X^*$ is cont. differentiable

- $\tilde{E}(u, 0) = E(u, \Omega), \qquad u \in X$
- $E(u_t, \Omega_t) = 0 \Leftrightarrow \tilde{E}(u^t, t) = 0$

(H2) $\tilde{E}(u^t, t) = 0$ has unique solution and

$$\lim_{t \to 0} \frac{|u^t - u^0|_X}{t^{1/2}} = 0.$$

(H3) $E_u(u, \Omega) \in \mathcal{L}(X, X^*)$ satisfies

$$\langle E(v, \Omega) - E(u, \Omega) - E_u(u, \Omega)(v - u), \psi \rangle_{X^* \times X} = \mathcal{O}(|v - u|_X^2)$$

(H4) $\tilde{E}$ and E satisfy

$$\lim_{t \to 0} \frac{1}{t} \langle \tilde{E}(u^t, t) - \tilde{E}(u, t) - E(u^t, \Omega) + E(u, \Omega), \psi \rangle_{X^* \times X} = 0$$

for $\psi \in X$ and $E(u, \Omega) = \tilde{E}(u^t, t) = 0$.

(H5) $j_1, j_2 \in C^{1,1}$

Theorem (IKP,'08) Assume (H1) – (H5), that the adjoint equation

$$\langle E_u(u,\Omega)\psi, p\rangle = (j_1'(u),\psi)_\Omega + (j_2'(u),\psi)_\Gamma$$

has a unique solution p where u is given by

$$E(u,\Omega) = \tilde{E}(u,0) = 0.$$

Then

$$\begin{aligned} dJ(u,\Omega)h = &-\frac{d}{dt}\langle \tilde{E}(u,t), p\rangle_{X\times X^*}|_{t=0} \\ &+ \int_\Omega j_1(u)\,\mathrm{div}\,h\,dx + \int_\Gamma j_2(u)\,\mathrm{div}_\Gamma\,h\,dx. \end{aligned}$$

Comments:

- $X \hookrightarrow L^2(\Omega)$
- (H4) amounts to regularity assumptions on coefficients of PDE and h
- (H5) implies

$$\|j_i(v) - j_i(u) - j'(u)(v-u)\|_{L^1(\Omega)} \leq c\|v-u\|_X^2$$

- Hölder continuity in (H2) follows from

$$\langle E_u(u,\Omega)\delta u, \psi\rangle = \langle \tilde{E}_u(u,0)\delta u, \psi\rangle = \langle f, \psi\rangle$$

admits a unique solution $\delta u \in X$ for all $f \in X^*$.

Outline of proof

$$\delta_t = \det DF_t.$$

Consider

$$\begin{aligned}
&\frac{1}{t}(J(u_t,\Omega_t) - J(u,\Omega))\\
&\quad = \frac{1}{t}\Big(\int_{\Omega_t} j_1(u_t)\,dx - \int_{\Omega} j_1(u)\,dx\Big)\\
&\quad = \frac{1}{t}\int_{\Omega} (\delta_t j_1(u^t) - j_1(u))\,dx\\
&\quad = \frac{1}{t}\int_{\Omega} \delta_t\big(j_1(u^t) - j_1(u) - j_1'(u)(u^t-u)\big)\,dx\\
&\quad + \int_{\Omega}\frac{1}{t}(\delta_t - 1)j_1'(u)(u^t-u)\,dx + \int_{\Omega}\frac{1}{t}(\delta_t-1)j_1(u)\,dx\\
&\quad + \frac{1}{t}\int_{\Omega} j_1'(u)(u^t-u)\,dx.
\end{aligned}$$

Outline of proof

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Use the adjoint equation for the last term

$$\int_\Omega j_1'(u)(u^t-u)\,dx = \langle E_u(u,\Omega)(u^t-u),p\rangle_{X\times X^*}$$
$$= -\langle E(u^t,\Omega)-E(u,\Omega)-E_u(u,\Omega)(u^t-u),p\rangle$$
$$-\langle \tilde{E}(u^t,t)-\tilde{E}(u,t)-E(u^t,\Omega)+E(u,\Omega),p\rangle$$
$$-\langle \tilde{E}(u,t)-\tilde{E}(u,0),p\rangle$$

implies

$$\lim_{t\to 0}\frac{1}{t}\int_\Omega j_1'(u)(u^t-u)\,dx = -\frac{d}{dt}\langle \tilde{E}(u,t),p\rangle_{X\times X^*}\big|_{t=0}.$$

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