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Shape optimization and free boundary problems

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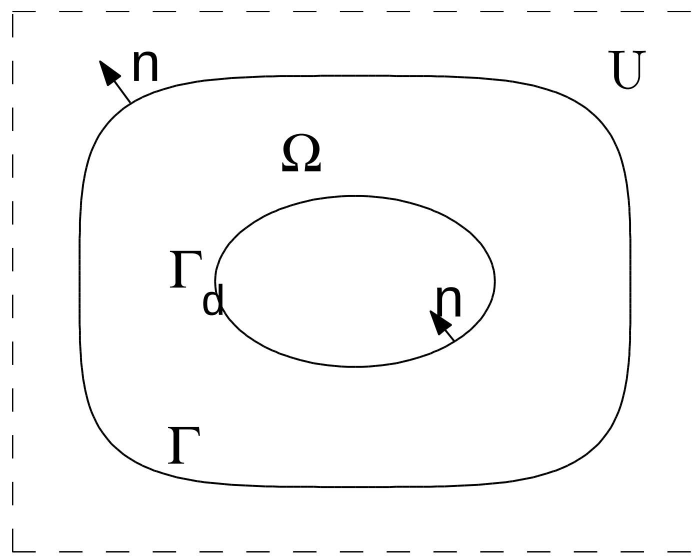
- Free boundary problems
- Numerical approaches
- Some basic concepts in shape optimization
- Material derivatives of the states
- Shape gradient of J

The exterior Bernoulli problem

Find (Ω, u) s.th.

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= 1 \quad \text{on } \Gamma_f, \\ u &= 0 \quad \text{on } \Gamma, \\ \frac{\partial u}{\partial \nu} &= \lambda \quad \text{on } \Gamma, \end{aligned}$$

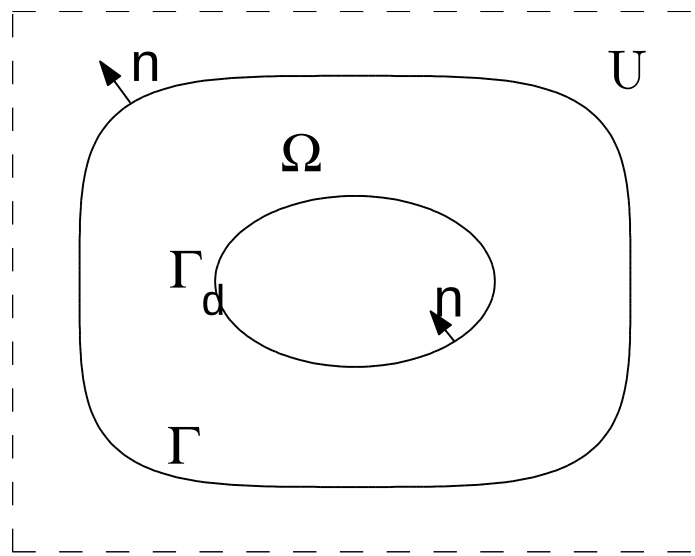


Ref.: Flucher and Rumpf, 1997

A Stokes free problem

Find (Ω, u) s.th.

$$
\begin{aligned}
&-2\,\mathrm{div}\,\sigma(u) + \nabla p = f && \text{in } \Omega,\\
&\mathrm{div}\,u = 0 && \text{in } \Omega,\\
&u = 0 && \text{on } \Gamma_f,\\
&u \cdot \nu = 0 && \text{on } \Gamma,\\
&\nu^T \sigma(u)\tau = 0 && \text{on } \Gamma,\\
&2\,\nu^T \sigma(u)\nu - p = \lambda && \text{on } \Gamma.
\end{aligned}
$$



where

$$\sigma(u) = \frac{1}{2}(\nabla u + \nabla u^T)$$

Ref.: Bouchon,Sayeh, Touzani,P.

Trial methods:

- Choose an initial guess for the free boundary
- Discard one of the BC on the free boundary
- Solve the resulting BVP
- Use the discarded BC to update the free boundary Γ_k by

$$\Gamma_{k+1} = \Gamma_k + s_k \nu_k$$

Example: Given Ω_k solve

$$\begin{aligned} -\Delta u_k &= 0 && \text{in } \Omega_k, \\ u_k &= 1 && \text{on } \Gamma_f, \\ \frac{\partial u_k}{\partial \nu_k} &= \lambda && \text{on } \Gamma_k, \end{aligned}$$

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Choose Γ_{k+1} s.th. $u|_{\Gamma_{k+1}} = 0$, i.e.

$$0 = u_k(x_{k+1}) = u_k(x_k) + \frac{\partial u_k}{\partial \nu_k}(x_k) s_k = u_k(x_k) + \lambda s_k$$

$$\Rightarrow s_k = -\frac{u_k}{\lambda}, \qquad \text{i.e}$$

$$\Gamma_{k+1} = \Gamma_k - \frac{u_k}{\lambda}$$

Ref.: Flucher and Rumpf, 1997;
Tihonen/Järvinen 1992;
Bouchon/Clain/Touzani 2005, 2008

Shape optimization approach

Neumann formulation

$$\min_{\Gamma} J(\Gamma) \equiv \frac{1}{2} \int_{\Gamma} u_N^2 \, d\sigma$$

subject to:

$$-\Delta u_N = 0 \qquad \text{in } \Omega,$$

$$u_N = 1 \qquad \text{on } \Gamma_f$$

$$\frac{\partial u_N}{\partial \nu} = \lambda \qquad \text{on } \Gamma$$

Ref.: Tihonen, 1997

Ito, Kunisch, P., 2006, 2008

Dirichlet formulation

$$\min_{\Gamma} J(\Gamma) \equiv \frac{1}{2}\int_{\Gamma} (\frac{\partial u_D}{\partial \nu} - \lambda)^2 d\sigma$$

subject to:

$$-\Delta u_D = 0 \qquad \text{in } \Omega$$
$$u_D = 1 \qquad \text{on } \Gamma_f$$
$$u_D = 0 \qquad \text{on } \Gamma$$

Analysis using embedding domain techniques in:

Haslinger-Kunisch-Kozubek-P: 2004, 2003

A Dirichlet-Neumann approach

$$\min_{\Omega} J(\Omega) = \frac{1}{2}\int_{\Omega} |\nabla u_D - \nabla u_N|^2$$

where

$$\begin{aligned} -\Delta u_D &= 0, & -\Delta u_N &= 0, & \text{in } \Omega \\ u_D &= 1, & u_N &= 1, & \text{on } \Gamma_f, \\ u_D &= 0, & \frac{\partial u_N}{\partial \nu} &= \lambda & \text{on } \Gamma, \end{aligned}$$

Ref: Ben Abda-Bouchon-Touzani-Sayeh-P., 2013

Eppler-Harbrecht, 2009

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Find the shape of the excitation part of an acoustic lens such that a prescribed pressure distribution y_d is achieved, i.e.

$$\min J(\Omega) \equiv \int_\Omega |\rho\psi' - y_d|^2$$

subject to Westervelt equation

$$((1 - k\psi')\psi')' - c^2\Delta\psi - b\Delta(\psi') = 0$$

+ BC

The Eulerian derivative of J:

Construct a family of perturbations Ω_t of a reference domain $\Omega \in C^{1,1}$, $\bar{\Omega} \subset U$, by perturbing the identity:

$$\mathcal{D} = \{h \in C^{1,1}(\bar{U})^2 \colon h|_{\partial U} = h|_{\Gamma_f} = 0\}.$$

For $h \in \mathcal{D}$ and t sufficiently small define

$$F_t = id + th,$$
$$\Omega_t = F_t(\Omega), \qquad \Gamma_t = F_t(\Gamma).$$

Then F_t is a $C^{1,1}$-diffeomorphism which preserves the regularity of Ω.

Define

$$dJ(u,\Omega)h = \lim_{t\to 0} \frac{1}{t}(J(u_t,\Omega_t) - J(u,\Omega)),$$

where u and u_t satisfy

$$E(u,\Omega) = 0, \qquad E(u_t,\Omega_t) = 0.$$

$dJ(u,\Omega)$ is called shape derivative of J if $dJ(u,\Omega)h$ exists for all $h \in \mathfrak{D}$ and $dJ(u,\Omega) \in \mathfrak{D}^*$.

Delfour-Hadamard-Zolesio structure theorem:

$$dJ(u,\Omega)h = \int_\Gamma G(s) h \cdot \nu \, ds.$$

Method of mapping: Let

$$\varphi_t \colon \Omega_t \to \mathbb{R}^2$$

then

$$\varphi^t = \varphi_t \circ F_t \colon \Omega \to \mathbb{R}^2, \qquad \varphi_0 = \varphi^0 \colon \Omega \to \mathbb{R}^2$$

and

$$\dot{\varphi}(h) = \lim_{t \to 0} \frac{1}{t}(\varphi_t \circ F_t - \varphi)$$

is called material derivative of φ_t in direction h

$$\varphi'(h) = \lim_{t \to 0} \frac{1}{t}(\varphi_t - \varphi)$$

is called local (shape) derivative of φ. They are related by

$$\dot{\varphi}(h) = \varphi'(h) + \nabla\varphi \cdot h.$$

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Lemma. 1. $\varphi_t \in H^1(\Omega_t)$ *iff* $\varphi^t \in H^1(\Omega)$ *and*

$$(\nabla \varphi_t) \circ F_t = M_t \nabla \varphi^t,$$

with $M_t = DF_t^{-T}$.

2. *If* $\varphi_t \in L^1(\Omega_t)$*, Then* $\varphi^t \in L^1(\Omega)$ *and*

$$\int_{\Omega_t} \varphi_t = \int_{\Omega} \delta_t \varphi^t, \tag{1}$$

where $\delta_t = \det(DF_t) = \det(I + t\,\nabla h^{\mathsf{T}})$.

3. *If* $\varphi_t \in L^1(\Gamma_t)$*, then* $\varphi^t \in L^1(\Gamma)$ *and we have*

$$\int_{\Gamma_t} \varphi_t = \int_{\Gamma} \omega_t \varphi^t, \tag{2}$$

with $\omega_t = \delta_t\,|M_t \nu|$.

How does F_t affect the normal field?

ν_t ... normal to Γ_t

b_t ... signed distance function to Γ_t, hence

$$\nu_t = \nabla b_t.$$

Set

$$b^t = b_t \circ F_t, \qquad \nu^t = \nu_t \circ F_t.$$

Since $F_t(x) \in \Gamma_t$ iff $x \in \Gamma$

$$b^t(x) = b_t(F_t(x)) = 0 \quad \text{iff } x \in \Gamma,$$

i.e. Γ is zero-level set of b^t, hence

$$\nabla b^t = \alpha\nu, \quad \text{for some } \alpha > 0.$$

Thus

$$\nu^t = \nabla b_t \circ F_t = M_t \nabla b^t = \alpha M_t \nu$$

By $|\nu^t| = 1$

$$\nu^t = \frac{M_t \nu}{|M_t \nu|}.$$

Note

$$\lim_{t \to 0} \delta_t = \delta_0 = 1, \qquad \lim_{t \to 0} \omega_t = \omega_0 = 1,$$
$$\lim_{t \to 0} M_t = M_0 = I,$$

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$$\min_{\Omega} J(\Omega) = \frac{1}{2}\int_{\Omega} |\nabla u_D - \nabla u_N|^2$$

where

$$\begin{aligned} -\Delta u_D &= 0, & -\Delta u_N &= 0, & &\text{in } \Omega \\ u_D &= 1, & u_N &= 1, & &\text{on } \Gamma_f, \\ u_D &= 0, & \frac{\partial u_N}{\partial \nu} &= \lambda & &\text{on } \Gamma, \end{aligned}$$

Find u_D, $u_N \in H^1(\Omega)$ s. th. $u_D = u_N = 1$ on Γ_f, $u_D = 0$ on Γ

$$\begin{aligned} (\nabla u_D, \nabla v) &= 0, & & v \in H_0^1(\Omega), \\ (\nabla u_N, \nabla v) &= \int_{\Gamma} \lambda v, & & v \in V = \{v \in H^1(\Omega) \colon v = 0 \text{ on } \Gamma_f\}. \end{aligned}$$

by elliptic regularity: u_D, $u_N \in H^2(\Omega)$

embed into family of problems on deformed domains

$$u_{D,t}, \quad u_{N,t} \ldots \Omega_t = F_t(\Omega)$$
$$(u_{D,0} = u_D, \quad u_{N,0} = u_N)$$

refer $u_{D,t}$, $u_{N,t}$ to the reference domain Ω

$$u_D^t = u_{D,t} \circ F_t, \qquad u_N^t = u_{N,t} \circ F_t.$$

In view of $\nabla \varphi_t \circ F_t = M_t \nabla \varphi^t$ one obtains

$$\int_{\Omega_t} \nabla u_{N,t} \cdot \nabla \varphi_t = \int_\Omega \delta_t M_t \nabla u_N^t \cdot M_t \nabla \varphi^t = \int_\Omega \delta_t M_t^T M_t \nabla u_N^t \cdot \nabla \varphi^t,$$
$$\int_{\Gamma_t} \lambda \varphi_t = \int_\Gamma \lambda \omega_t \varphi^t$$

Perturbed equations on reference domain:

Find u_D^t, $u_N^t \in H^1(\Omega)$ such that $u_D^t = u_N^t = 1$ on Γ_f , $u_D^t = 0$ on Γ

$$(A_t \nabla u_D^t, \nabla v) = 0, \qquad v \in H_0^1(\Omega),$$

$$(A_t \nabla u_N^t, \nabla v) = \lambda \int_\Gamma \omega_t v, \qquad v \in V$$

where

$$A_t = \delta_t M_t^T M_t$$

Define the bilinear form

$$a^t : V \times V \to \mathbb{R}$$
$$a^t(u, v) = \int_\Omega A_t \nabla u \cdot \nabla v$$

It follows from $\lim_{t\to 0} A_t = I$ and

$$a^t(u, u) = \int_\Omega \nabla u \cdot \nabla u + \int_\Omega (A_t - I)\nabla u \cdot \nabla u$$

that a^t is coercive for $|t|$ sufficiently small.

Hence: perturbed problems have unique solutions u_D^t, u_N^t

material derivative of u_N^t:

by implicit function theorem:

Define $\Phi\colon I \times V \to V^*$, $I = (-t_h, t_h)$

$$\langle \Phi(t,y), v\rangle = \int_\Omega A_t \nabla(y+u_N)\cdot\nabla v - \lambda\int_\Gamma \omega_t v, \quad v \in V$$

Since

$$\langle \Phi(0,0), v\rangle = \int_\Omega \nabla u_N\cdot\nabla v - \lambda\int_\Gamma v = 0, \quad v \in V$$

one finds $\Phi(0,0) = 0$ and similarly

$$\Phi(t, u_N^t - u_N) = 0, \qquad t \in I$$

The mappings $t \mapsto \delta_t$, $t \mapsto M_t$ and $t \mapsto \omega_t$ with are $\mathcal{C}^1$ in a neighborhood of 0 and

$$\dot{\delta}_t = \frac{d\delta_t}{dt}\ |_{t=0} = \operatorname{div} h,$$

$$\dot{M}_t = \frac{dM_t}{dt}\ |_{t=0} = -\nabla h,$$

$$\dot{A}_t = \frac{dA_t}{dt}\ |_{t=0} = A = \operatorname{div} hI - \nabla h - (\nabla h)^T,$$

$$\dot{\omega}_t = \frac{d\omega_t}{dt}\ |_{t=0} = \operatorname{div}_\Gamma h = \operatorname{div} h|_\Gamma - \nabla h\nu \cdot \nu$$

Verify: Φ is continuously differentiable.

Furthermore,

$$\langle D_y \Phi(0,0)\delta y, v\rangle = \int_\Omega \nabla \delta y \cdot \nabla v$$

implies

$$D_y \Phi(0,0) \colon V \to V^* \qquad \text{is an isomorphism}$$

By implicit function theorem there exists $t \to y^t$, continuously differentiable on I such that

$$\Phi(t, y^t) = 0, \qquad t \in I$$

By uniqueness

$$y^t = u_N^t - u_N$$

Thus $t \to u_N^t$ is differentiable and $\dot{u}_N = \dot{y} = \frac{dy^t}{dt}\,|_{t=0}$ satisfies

$$\langle D_y\Phi(0,0)\dot{u}_N, v\rangle + \langle \frac{\partial \Phi}{\partial t}(0,0), v\rangle = 0, \qquad v \in V$$

equivalently

$$\int_\Omega \nabla \dot{u}_N \cdot \nabla v = -\int_\Omega A\nabla u_N \cdot \nabla v + \lambda \int_\Gamma v\,\mathrm{div}_\Gamma\, h, \qquad v \in V$$

Similarly

$$\int_\Omega \nabla \dot{u}_D \cdot \nabla v = -\int_\Omega A\nabla u_D \cdot \nabla v, \qquad v \in H_0^1(\Omega)$$

Lemma. *Let* u, $v \in H^2(\Omega)$ *then*

$$\int_\Omega A\nabla u \cdot \nabla v = \int_\Omega \big(\Delta u\,(h\cdot\nabla v) + \Delta v\,(h\cdot\nabla u)\big)$$

$$- \int_\Gamma \big(\frac{\partial u}{\partial \nu}\,(h\cdot\nabla v) + \frac{\partial v}{\partial \nu}\,(h\cdot\nabla u)\big) + \int_\Gamma \nabla u\cdot\nabla v\,(h\cdot\nu).$$

As a consequence

for $u = v = u_D$, hence $\nabla u_D = \frac{\partial u_D}{\partial \nu}\nu$

$$\int_\Omega A\nabla u_D\cdot\nabla u_D = -\int_\Gamma (\frac{\partial u_D}{\partial \nu})^2\,h\cdot\nu,$$

for $u = v = u_N$, hence $\frac{\partial u_N}{\partial \nu} = \lambda$

$$\int_\Omega A\nabla u_N\cdot\nabla u_N = -\int_\Gamma |\nabla u_N|^2\,h\cdot\nu - 2\lambda\int_\Gamma h\cdot\nabla u_N$$

Proof of Lemma.

$$\int_\Omega A\nabla u \cdot \nabla v = \int_\Omega \operatorname{div} h \nabla u \cdot \nabla v - \int_\Omega \nabla h \nabla u \cdot \nabla v - \int_\Omega (\nabla h)^T \nabla u \cdot \nabla v$$

Integrating by parts

$$\begin{aligned} \int_\Omega \operatorname{div} h \nabla u \cdot \nabla v &= - \int_\Omega h \cdot \nabla(\nabla u \cdot \nabla v) + \int_\Gamma (\nabla u \cdot \nabla v)\, h \cdot \nu \\ &= - \int_\Omega h \cdot (\nabla^2 u \nabla v + \nabla^2 v \nabla u) + \int_\Gamma (\nabla u \cdot \nabla v). \end{aligned}$$

Observe

$$\begin{aligned} \operatorname{div}\big(\nabla v (h \cdot \nabla u)\big) &= \Delta v (h \cdot \nabla u) + \nabla v \cdot \nabla (h \cdot \nabla u) \\ &= \Delta v (h \cdot \nabla u) + \nabla v \cdot \nabla h \nabla u + \nabla v \cdot \nabla^2 u h \end{aligned}$$

□

Therefore

$$-\int_\Omega \nabla h \nabla u \cdot \nabla v = -\int_\Omega \operatorname{div}\big(\nabla v(h\cdot\nabla u)\big) + \int_\Omega \big(\nabla v\cdot\nabla^2 u h + \Delta v(h\cdot\nabla u)\big)$$
$$= -\int_\Gamma \frac{\partial v}{\partial \nu}\,(h\cdot\nabla u) + \int_\Omega \big(\nabla^2 u \nabla v\cdot h + \Delta v(h\cdot\nabla u)\big)$$

By symmetry

$$-\int_\Omega (\nabla h)^T \nabla u\cdot\nabla v = -\int_\Gamma \frac{\partial u}{\partial \nu}\,(h\cdot\nabla v) + \int_\Omega \big(\nabla^2 v\nabla u\cdot h + \Delta u(h\cdot\nabla v)\big)$$

The result follows adding the integrals.

Shape derivative of J:

Recall

$$DJ(\Omega)h = \lim_{t\to 0}\frac{1}{t}(J(\Omega_t) - J(\Omega)) = \frac{d}{dt}J(\Omega_t)\mid_{t=0}$$

and

$$J(\Omega_t) = \frac{1}{2}\int_{\Omega_t}|\nabla u_{D,t} - \nabla u_{N,t}|^2 = \frac{1}{2}\int_{\Omega}\delta_t|M_t\nabla u_D^t - M_t\nabla u_N^t|^2$$
$$= \frac{1}{2}\int_{\Omega}A_t\nabla(u_D^t - u_N^t)\cdot\nabla(u_D^t - \nabla u_N^t)$$

and therefore

$$DJ(\Omega)h = \frac{1}{2}\int_{\Omega}A\nabla(u_D - u_N)\cdot\nabla(u_D - u_N)$$
$$+ \int_{\Omega}\nabla(\dot{u}_D - \dot{u}_N)\cdot\nabla(u_D - u_N)$$

Eliminate $\dot{u}_D$ and $\dot{u}_N$:

Since $\dot{u}_D \in H_0^1(\Omega)$

$$\int_\Omega \nabla u_D \cdot \nabla \dot{u}_D = 0,$$

$$\int_\Omega \nabla u_N \cdot \nabla \dot{u}_D = \lambda \int_\Gamma \dot{u}_D = 0,$$

therefore

$$\int_\Omega \nabla \dot{u}_D \cdot \nabla (u_D - u_N) = 0.$$

Since $u_D - u_N \in V$

$$\int_\Omega \nabla \dot{u}_N \cdot \nabla (u_D - u_N) = -\int_\Omega A \nabla u_N \cdot \nabla (u_D - u_N) + \lambda \int_\Gamma (u_D - u_N) \operatorname{div}_\Gamma h$$

Therefore

$$\int_\Omega \nabla(\dot{u}_D - \dot{u}_N)\cdot\nabla(u_D - u_N) = \int_\Omega A\nabla u_N\cdot\nabla(u_D - u_N) - \lambda\int_\Gamma (u_D - u_N)\,\mathrm{div}_\Gamma\, h$$

Hence

$$\begin{aligned} DJ(\Omega)h &= \frac{1}{2}\int_\Omega A\nabla(u_D - u_N)\cdot\nabla(u_D - u_N) + \int_\Omega A\nabla u_N\cdot\nabla(u_D - u_N) \\ &\quad + \lambda\int_\Gamma u_N\,\mathrm{div}_\Gamma\, h \\ &= \frac{1}{2}\int A\nabla u_D\cdot\nabla u_D - \frac{1}{2}\int_\Omega A\nabla u_N\cdot\nabla u_N + \lambda\int_\Gamma u_N\,\mathrm{div}_\Gamma\, h. \end{aligned}$$

By the Lemma

$$DJ(\Omega)h = -\frac{1}{2}\int_\Gamma \left(\left(\frac{\partial u_D}{\partial \nu}\right)^2 + |\nabla u_N|^2\right)h\cdot\nu + \lambda\int_\Gamma (h\cdot\nabla u_N + u_N\,\mathrm{div}_\Gamma\, h)$$

Recall tangential Green's formula

$$\int_\Gamma (\varphi \operatorname{div}_\Gamma h + \nabla_\Gamma \varphi \cdot h) = \int_\Gamma \kappa \varphi h \cdot \nu$$

where κ denotes the curvature of Γ and

$$\nabla_\Gamma \varphi = \nabla \varphi|_\Gamma - \frac{\partial \varphi}{\partial \nu} \nu$$

Hence

$$\int_\Gamma (h \cdot \nabla u_N + u_N \operatorname{div}_\Gamma h) = \int_\Gamma (\frac{\partial u_N}{\partial \nu} + u_N \kappa)\, h \cdot \nu$$

which entails

$$DJ(\Omega)h = \frac{1}{2} \int_\Gamma (\lambda^2 - (\frac{\partial u_D}{\partial \nu})^2 + 2\lambda \kappa u_N - (\frac{\partial u_N}{\partial \nu})^2\big)\, h \cdot \nu$$

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Recall: u_N, u_N^t satisfy

$$(\nabla u_N, \nabla v) = \lambda \int_\Gamma v$$
$$(A_t \nabla u_N^t, \nabla v) = \lambda \int_\Gamma \omega_t v, \qquad v \in V.$$

uniform boundedness of u_n^t:

$$(A_t \nabla (u_N^t - u_N), \nabla v) = \lambda \int_\Gamma \omega_t v - (A_t \nabla u_N, \nabla v), \quad v \in V.$$

Since $v = u_N^t - u_N \in V$ and A_t is uniformly coercive

$$\frac{1}{2} \|\nabla (u_N^t - u_N)\|^2 \leq |\lambda| \int_\Gamma \omega_t |u_N^t - u_N| + \left| \big(A_t \nabla u_N, \nabla (u_N^t - u_N) \big) \right|$$

a bound for $u_N^t - u_N$ follows.

Continuity: $\lim_{t\to 0} u_N^t = u_N$ Use the equations for u_N^t and u_N

$$(\nabla(u_N^t - u_N), \nabla v) = (A_t \nabla u_N^t, \nabla v) - \big((A_t - I)\nabla u_N^t, \nabla v\big) - (\nabla u_N, \nabla v)$$
$$= -(A_t - I)\nabla u_N^t), \nabla v) + \lambda \int_\Gamma (\omega_t - 1) v$$

Choose $v = u_N^t - u_N$

$$\|\nabla(u_N^t - u_N)\| \le \|A_t - I\|_{L^\infty} \|\nabla u_N^t\| + |\lambda| \, \|\omega_t - 1\|_{L^2} \|\tau_\Gamma\|$$

Material derivative of u_N^t: Choose a sequence $t_k \to 0$ and set

$$y_k = \frac{1}{t_k}(u_N^{t_k} - u_N).$$

Then

$$(\nabla y_k, \nabla v) = -(\frac{1}{t_k}(A_{t_k} - I)\nabla u_N^{t_k}, \nabla v) + \lambda \int_\Gamma \frac{1}{t_k}(\omega_{t_k} - 1)v \qquad (*)$$

As before (y_k) is bounded in V

$\Rightarrow$ subsequence (y_k) and $y \in V$ such that

$$y_k \rightharpoonup y$$

Passing to the limit y satisfies

$$(\nabla y, \nabla v) = -(A\nabla u_N, \nabla v) + \lambda \int_\Gamma v \operatorname{div}_\Gamma h$$

By uniqueness y is independent of (t_k)

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Choose $v = y_k$ in $(*)$

$$\|\nabla y_k\|^2 = -\big(\frac{1}{t_k}(A_{t_k} - I)\nabla u_N^{t_k}, \nabla y_k\big) + \lambda \int_\Gamma \frac{1}{t_k}(\omega_{t_k} - 1) y_k$$

trace operator $\tau_\Gamma \in \mathcal{L}(V, H^{1/2}(\Gamma))$ hence

$$\tau_\Gamma y_k \rightharpoonup \tau_\Gamma y \qquad \text{in } H^{1/2}(\Gamma),$$

thus by compact embedding strongly in $L^2(\Gamma)$.

Passing to the limit

$$\lim_{k\to\infty} \|\nabla y_k\|^2 = -(A\nabla u_N, \nabla y) + \lambda \int_\Gamma y \operatorname{div}_\Gamma h = \|y\|^2$$

This implies $y_k \to y$ strongly in V.