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1. ***Moser iteration technique and regularity.***
2. ***Boundary value problems involving a nonhomogeneous operator.***
3. ***Smooth minimizers versus Sobolev minimizers.***
4. ***Multiple solutions for nonlinear elliptic problems.***

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A general operator

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with Lipschitz boundary $\partial\Omega$, let $p \in (1, +\infty)$.

We consider a general operator $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying:

$\mathrm{H}(a)_1$ **(i)** $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous;

(ii) there is a constant $c_1 > 0$ such that

$$|a(x,\xi)| \leq c_1(1 + |\xi|^{p-1}) \text{ for all } x \in \overline{\Omega}, \text{ all } \xi \in \mathbb{R}^N;$$

(iii) there are constants $c_0 > 0$ and $R, \sigma \geq 0$ such that

$$(a(x,\xi),\xi)_{\mathbb{R}^N} \geq c_0(R + |\xi|)^{p-\sigma}|\xi|^{\sigma} \text{ for all } x \in \overline{\Omega}, \text{ all } \xi \in \mathbb{R}^N.$$

Examples

Many interesting operators fit the setting of hypotheses $\mathrm{H}(a)_1$:

(a) $a(x,\xi) = |\xi|^{p-2}\xi$, so that $\operatorname{div} a(x,\nabla u)$ is the p-Laplacian in this case.

If a_1 satisfies $\mathrm{H}(a)_1$ and a_2 satisfies $\mathrm{H}(a)_1$(i), (ii), and $(a_2(x,\xi),\xi)_{\mathbb{R}^N} \geq 0$, then $a_1 + a_2$ also satisfies $\mathrm{H}(a)_1$. Thus, we can derive other examples from (a):

(b) $a(x,\xi) = |\xi|^{p-2}\xi + |\xi|^{q-2}\xi$ with $1 < q \leq p$ (the (p,q)-Laplacian case);

(c) $a(x,\xi) = |\xi|^{p-2}\xi + \ln(1+|\xi|^{p-2})\xi$;

(d) $a(x,\xi) = |\xi|^{p-2}\xi + c\,\frac{|\xi|^{p-2}\xi}{1+|\xi|^p}$ with $c > 0$ (in this case $\operatorname{div} a(x,\nabla u)$ is the sum of the p-Laplacian and a generalized mean curvature operator).

In each example (a)–(d), the map a satisfies $\mathrm{H}(a)_1$(iii) with $R = 0$.

If a satisfies $\mathrm{H}(a)_1$, then so does $(x,\xi) \mapsto \theta(x)a(x,\xi)$ whenever $\theta \in C(\overline{\Omega},(0,+\infty))$.

Thus, further examples can be derived from (a)–(d).

p^* denotes the critical exponent of p, i.e.,

$$p^* = \frac{Np}{N-p} \quad \text{if } N > p \quad \text{and} \quad p^* = +\infty \quad \text{if } N \leq p.$$

$\mathrm{H}(f)$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function

(i.e., $f(\cdot, s)$ is measurable for all $s \in \mathbb{R}$ and $f(x, \cdot)$ is continuous for a.a. $x \in \Omega$)

and there are constants $c > 0$ and $r \in (p, p^*)$ such that

$$|f(x,s)| \leq c(1 + |s|^{r-1}) \text{ for a.a. } x \in \Omega, \text{ all } s \in \mathbb{R}.$$

Hypothesis $\mathrm{H}(f)$ guarantees that the Nemytskii operator

$$N_f : W^{1,p}(\Omega) \to L^{r'}(\Omega), \quad u \mapsto f(\cdot, u(\cdot)),$$

is well defined and continuous. Thus, $N_f(u) \in (W^{1,p}(\Omega))^*$ whenever $u \in W^{1,p}(\Omega)$.

Given the operator a and the function f, we consider nonlinear elliptic problems, under Dirichlet boundary condition:

$$(1) \qquad \begin{cases} -\mathrm{div}\, a(x, \nabla u(x)) = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and under Neumann boundary condition:

$$(2) \qquad \begin{cases} -\mathrm{div}\, a(x, \nabla u(x)) = f(x, u(x)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} = 0 & \text{on } \partial\Omega. \end{cases}$$

$\mathrm{H}(a)_1$(ii) implies $a(\cdot, \nabla u(\cdot)) \in L^{p'}(\Omega, \mathbb{R}^N)$ whenever $u \in W^{1,p}(\Omega)$, where $p' = \frac{p}{p-1}$.
In particular, the divergence $\mathrm{div}\, a(x, \nabla u)$ (in the distributional sense) is well defined.

In (2), we denote $\frac{\partial u}{\partial n_a} = \gamma_n(a(\cdot, \nabla u(\cdot)))$, where γ_n is the generalized normal derivative.

Definition. (a) A weak solution of problem (1) is a function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega (a(x, \nabla u), \nabla v)_{\mathbb{R}^N} \, dx = \int_\Omega f(x, u(x)) \, v(x) \, dx \text{ for all } v \in W_0^{1,p}(\Omega).$$

(b) A weak solution of problem (2) is a function $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega (a(x, \nabla u), \nabla v)_{\mathbb{R}^N} \, dx = \int_\Omega f(x, u(x)) \, v(x) \, dx \text{ for all } v \in W^{1,p}(\Omega).$$

Remark 1. If u is a weak solution of (2), then $u \in L^\infty(\Omega)$ (by Moser iteration technique), so $f(\cdot, u(\cdot)) \in L^\infty(\Omega)$ (by $\mathrm{H}(f)$).
Since $-\mathrm{div}\, a(x, \nabla u) = f(x, u)$ (in distributions), we get $-\mathrm{div}\, a(\cdot, \nabla u(\cdot)) \in L^\infty(\Omega)$.
Then Definition (b) implies $\frac{\partial u}{\partial n_a} = 0$ (thanks to nonsmooth Green's identity).
Thereby, u satisfies the boundary condition in problem (2).

Moser iteration technique

Theorem 1. *Assume* $\mathrm{H}(a)_1$ *and* $\mathrm{H}(f)$*. Let* $u \in W^{1,p}(\Omega)$ *be such that*

$$\int_\Omega (a(x, \nabla u), \nabla v)_{\mathbb{R}^N}\, dx \le \int_\Omega f(x, u(x))\, v(x)\, dx$$

whenever $v = \min\{u^+, \lambda\}^\alpha$ *or* $v = -\min\{u^-, \lambda\}^\alpha$*, for* $\lambda > 0$ *and* $\alpha \ge 1$*.*
Then, $u \in L^\infty(\Omega)$ *and, for all real number* $\theta \in (r, p^*]$*, we have*

$$\|u\|_\infty \le M(1 + \|u\|_\theta)^{\frac{\theta - p}{\theta - r}}$$

where $M > 0$ *depends only on* c_0*,* c*,* R*,* σ*,* Ω*,* p*,* θ*, and* N*.*

Theorem 1 applies to weak solutions of problems (1) and (2).

By Sobolev embedding theorem, there is a constant $M_0 = M_0(\Omega, p, \theta, N) > 0$ such that

(3) $$\|v\|_\theta \leq M_0(\|v\|_p + \|\nabla v\|_p) \text{ for all } v \in W^{1,p}(\Omega).$$

We denote $v = u^+$.

Claim 1: There exists $M_1 = M_1(c_0, c, R, \sigma, \Omega, p, \theta, N) > 0$ such that, for all $\ell \in [0, +\infty)$,

$$1 + \int_\Omega v^{\theta(\ell+1)}\, dx \leq M_1(\ell+1)^\theta \left(1 + \int_\Omega v^{p\ell+r}\, dx\right)^{\frac{\theta}{p}}$$

(where the integrals may be infinite a priori).

Fix $\ell \in [0, +\infty)$ and $\lambda \in (0, +\infty)$, and denote $v_\lambda = \min\{v, \lambda\}$.

Towards the conclusion of Claim 1, by (3) we have

$$\int_\Omega v_\lambda^{\theta(\ell+1)}\, dx \leq M_0^\theta(\|v_\lambda^{\ell+1}\|_p + \|\nabla(v_\lambda^{\ell+1})\|_p)^\theta.$$

Now, it is enough to prove that (with $\widetilde{M}_1$ as M_1)

$$\|v_\lambda^{\ell+1}\|_p^\theta, \quad \|\nabla(v_\lambda^{\ell+1})\|_p^\theta \leq \widetilde{M}_1(\ell+1)^\theta \left(1 + \int_\Omega v^{p\ell+r}\, dx\right)^{\frac{\theta}{p}}.$$

Since $r \geq p$, for $M_2 = \max\{1, |\Omega|_N\}$, we have

$$\int_\Omega |v_\lambda^{\ell+1}|^p\, dx = \int_\Omega v_\lambda^{p\ell+p}\, dx \leq \int_\Omega (1 + v_\lambda^{p\ell+r})\, dx \leq M_2\left(1 + \int_\Omega v^{p\ell+r}\, dx\right).$$

Proof: Claim 1 (continue)

For the second needed inequality: the assumption, the fact that $v_\lambda = 0$ a.e. on $\{x \in \Omega : \; u(x) \le 0\}$, and $\mathrm{H}(f)$ yield (for some $M_3 = M_3(c, |\Omega|_N) > 0$)

$$\int_\Omega (a(x,\nabla u), \nabla(v_\lambda^{p\ell+1}))_{\mathbb{R}^N}\, dx \le \int_\Omega f(x,u) v_\lambda^{p\ell+1}\, dx$$

$$= \int_\Omega f(x,v) v_\lambda^{p\ell+1}\, dx \le \int_\Omega c(1+v^{r-1}) v^{p\ell+1}\, dx \le M_3\Big(1+\int_\Omega v^{p\ell+r}\, dx\Big).$$

Since v_λ is constant (so ∇v_λ vanishes) on $\{x \in \Omega : u(x) \ne v_\lambda(x)\}$, by $\mathrm{H}(a)_2$ (iii),

$$\int_\Omega (a(x,\nabla u), \nabla(v_\lambda^{p\ell+1}))_{\mathbb{R}^N}\, dx = (p\ell+1)\int_\Omega (a(x,\nabla u), \nabla v_\lambda)_{\mathbb{R}^N}\, v_\lambda^{p\ell}\, dx$$

$$= (p\ell+1)\int_\Omega (a(x,\nabla v_\lambda), \nabla v_\lambda)_{\mathbb{R}^N}\, v_\lambda^{p\ell}\, dx \ge c_0 \int_\Omega (R+|\nabla v_\lambda|)^{p-\sigma} |\nabla v_\lambda|^\sigma v_\lambda^{p\ell}\, dx.$$

We set (with $R, \sigma \ge 0$ from $\mathrm{H}(a)_1$ (iii)):

$$M_2 = M_2(R,\sigma,p) := \inf_{t\in[1,+\infty)} \Big(\tfrac{t}{R+t}\Big)^{\sigma-p} > 0.$$

Combining these inequalities, we conclude (where $M_4 = \max\{1, |\Omega|_N\} + \frac{M_3}{c_0 M_2}$)

$$\frac{1}{(\ell+1)^p}\int_\Omega |\nabla(v_\lambda^{\ell+1})|^p\, dx = \int_\Omega |\nabla v_\lambda|^p v_\lambda^{p\ell}\, dx$$

$$\le \int_{\{|\nabla v_\lambda|<1\}} v_\lambda^{p\ell}\, dx + \frac{1}{M_2}\int_{\{|\nabla v_\lambda|\ge 1\}} (R+|\nabla v_\lambda|)^{p-\sigma} |\nabla v_\lambda|^\sigma v_\lambda^{p\ell}\, dx \le M_4\Big(1+\int_\Omega v^{p\ell+r}\, dx\Big).$$

Proof: iteration and Claim 2

Since $p \le r < \theta$, we have $\alpha := \frac{r-p}{\theta-p}\,\theta \in [0, \theta)$.

Let $\{q_i\}_{i\ge 0} \subset [\theta, +\infty)$ be defined by

$$(4) \qquad q_i = \alpha + \left(\frac{\theta}{p}\right)^i (\theta - \alpha).$$

Thus, $\{q_i\}_{i\ge 0}$ is increasing and $q_i \to +\infty$ as $i \to \infty$. We have the inductive relation

$$q_{i+1} = \frac{\theta}{p}(q_i - r + p) \ \text{ for all } i \ge 0.$$

For $i \ge 0$, set $\ell_i = \frac{q_i - r}{p} \in (0, +\infty)$. Thus, $q_i = p\ell_i + r$ and $q_{i+1} = \theta(\ell_i + 1)$.

Applying Claim 1 with $\ell = \ell_i$, we obtain

$$(5) \qquad 1 + \int_\Omega v^{q_{i+1}}\, dx \le \frac{M_1}{p^\theta}\,(q_i - r + p)^\theta \left(1 + \int_\Omega v^{q_i}\, dx\right)^{\frac{\theta}{p}} \ \text{ for all } i \ge 0.$$

On the basis of (5), we will show:

Claim 2: There is $M_5 = M_5(c_0, c, R, \sigma, \Omega, p, \theta, N) > 0$ such that

$$J_i := \left(1 + \int_\Omega v^{q_i}\, dx\right)^{\frac{1}{q_i}} \le M_5(1 + \|u\|_\theta)^{\frac{\theta-p}{\theta-r}} \ \text{ for all } i \ge 0.$$

First, since $q_0 = \theta$, we have $J_0 < +\infty$ (see (3)) and, from (5), $\{J_i\}_{i\geq 0} \subset (0, +\infty)$.
Let $S_i = q_i \ln J_i$ and $M_6 = \left(\frac{M_1}{p^\theta}\right)^{\frac{1}{\theta}}$. Then, passing to $\log$ in (5), we have

$$S_{i+1} \leq \theta \ln M_6 + \theta \ln(q_i - r + p) + \frac{\theta}{p} S_i \leq \theta \ln M_6 + \theta \ln q_i + \frac{\theta}{p} S_i \text{ for all } i \geq 0.$$

Clearly, $q_i \leq \left(\frac{\theta}{p}\right)^i \theta$ (see (4)). Whence (for some $M_9 = M_9(c_0, c, R, \sigma, \Omega, p, \theta, N) > 0$)

$$S_{i+1} \leq (i+1) M_9 + \frac{\theta}{p} S_i \text{ for all } i \geq 0.$$

By induction, we derive (for some $M_{10} = M_{10}(c_0, c, R, \sigma, \Omega, p, \theta, N) > 0$)

$$\begin{aligned} S_i \; &\leq \; M_9 \sum_{j=0}^{i-1} (i-j)\left(\frac{\theta}{p}\right)^j + \left(\frac{\theta}{p}\right)^i S_0 = M_9 \frac{\left(\frac{\theta}{p}\right)^{i+1} - (i+1)\frac{\theta}{p} + i}{\left(\frac{\theta}{p} - 1\right)^2} + \left(\frac{\theta}{p}\right)^i S_0 \\ &\leq \left(\frac{\theta}{p}\right)^i \left(M_9 \frac{\theta}{p}\left(\frac{\theta}{p} - 1\right)^{-2} + \ln(1 + \|u\|_\theta^\theta)\right) \leq \left(\frac{\theta}{p}\right)^i \ln\left(M_{10}(1 + \|u\|_\theta)^\theta\right) \text{ for all } i \geq 0. \end{aligned}$$

Combining with (4), we get

$$\ln J_i = \frac{1}{q_i} S_i \leq \frac{\left(\frac{\theta}{p}\right)^i \ln\left(M_{10}(1 + \|u\|_\theta)^\theta\right)}{\alpha + \left(\frac{\theta}{p}\right)^i (\theta - \alpha)} \leq \frac{\ln\left(M_{10}(1 + \|u\|_\theta)^\theta\right)}{\theta - \alpha},$$

hence $J_i \leq M_5 \left(1 + \|u\|_\theta\right)^{\frac{\theta - p}{\theta - r}}$ for all $i \geq 0$, with $M_5 = M_{10}^{\frac{\theta - p}{\theta(\theta - r)}}$, whence Claim 2.

Claim 3: $v \in L^\infty(\Omega)$ and $\|v\|_\infty \le \widehat{M}_u := M_5\big(1 + \|u\|_\theta\big)^{\frac{\theta-p}{\theta-r}}$.

Let $w \in L^1(\Omega)$. For $i \ge 0$, we define $w_i \in L^{q_i'}(\Omega)$ by

$$w_i(x) = \begin{cases} |w(x)| & \text{if } |w(x)| < 1, \\ |w(x)|^{\frac{1}{q_i'}} & \text{if } |w(x)| \ge 1. \end{cases}$$

Since the sequence $\{q_i\}_{i\ge 0} \subset (1,+\infty)$ is increasing and $q_i \to +\infty$ as $i \to \infty$, the conjugate sequence $\{q_i'\}_{i\ge 0} \subset (1,+\infty)$ is decreasing and $q_i' \to 1$ as $i \to \infty$.

Hence, the sequence of functions $\{w_i\}_{i\ge 0}$ is nondecreasing and, for all $x \in \Omega$, we have $w_i(x) \to |w(x)|$ as $i \to \infty$. For all $i \ge 0$, we see that

$$\int_\Omega v w_i \, dx \le \|v\|_{q_i} \|w_i\|_{q_i'} \le \widehat{M}_u \Big(\int_{\{|w|<1\}} |w|^{q_i'} \, dx + \int_{\{|w|\ge 1\}} |w| \, dx \Big)^{\frac{1}{q_i'}} \le \widehat{M}_u \Big(\int_\Omega |w| \, dx \Big)^{\frac{1}{q_i'}}$$

(since $q_i' > 1$). Passing to the limit as $i \to \infty$, by invoking the Beppo Levi monotone convergence theorem, we obtain

$$\int_\Omega |vw| \, dx \le \widehat{M}_u \|w\|_1 \quad \text{for all } w \in L^1(\Omega).$$

Hence the map $\psi : w \mapsto \int_\Omega vw \, dx$ belongs to $(L^1(\Omega))^*$ and satisfies $\|\psi\| \le \widehat{M}_u$.

Therefore, $v \in L^\infty(\Omega)$ and $\|v\|_\infty \le \widehat{M}_u$. This shows Claim 3.

Proof: conclusion

By Claim 3, we obtain

$$\|u^+\|_\infty \leq M_5 \big(1 + \|u\|_\theta\big)^{\frac{\theta - p}{\theta - r}}.$$

Applying the first part of the proof to the maps

$\tilde{a}(x, \xi) := -a(x, -\xi)$, $\tilde{f}(x, s) := -f(x, -s)$, and to the function $\tilde{u} := -u$ instead of u,

we also derive $u^- \in L^\infty(\Omega)$ and

$$\|u^-\|_\infty \leq M_5 \big(1 + \|u\|_\theta\big)^{\frac{\theta - p}{\theta - r}}.$$

The proof of the theorem is then complete.

Regularity of solutions. Hypotheses on a

Hereafter, the boundary of Ω is assumed to be of class C^2.

$\mathrm{H}(a)_2$ **(i)** $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous, C^1 on $\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\})$, and $a(x,0) = 0$ for all $x \in \overline{\Omega}$;

(ii) there are constants $R \geq 0$, $c_0 > 0$ such that, for all $x \in \overline{\Omega}$, $\xi, \eta \in \mathbb{R}^N$, $\xi \neq 0$,

$$(a'_\xi(x,\xi)\eta, \eta)_{\mathbb{R}^N} \geq c_0(R + |\xi|)^{p-2}|\eta|^2;$$

(iii) there is a constant $c_1 > 0$ such that, for all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$, $\xi \neq 0$, we have

$$\|a'_\xi(x,\xi)\| \leq c_1(R + |\xi|)^{p-2}.$$

(iv) there is a constant $\alpha \in (0,1)$ such that, for all $x, y \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$, we have

$$|a(x,\xi) - a(y,\xi)| \leq c_1|x - y|^\alpha(1 + |\xi|^{p-2})|\xi|.$$

Regularity of solutions. Hypotheses on a

Remark 2. $\mathrm{H}(a)_2$ is stronger than $\mathrm{H}(a)_1$. Indeed, for all $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^N$, we obtain

$$\begin{aligned}(a(x,\xi),\xi)_{\mathbb{R}^N} &= \int_0^1 (a'_\xi(x,t\xi)\xi,\xi)_{\mathbb{R}^N}\,dt \geq c_0 \int_0^1 t^{p-2}(R+|\xi|)^{p-2}|\xi|^2\,dt \\ &= \frac{c_0}{p-1}\,(R+|\xi|)^{p-2}|\xi|^2,\end{aligned}$$

$$\begin{aligned}|a(x,\xi)| &= \int_0^1 \frac{(a(x,t\xi),a'_\xi(x,t\xi)\xi)_{\mathbb{R}^N}}{|a(x,t\xi)|}\,dt \leq c_1 \int_0^1 t^{p-2}(R+|\xi|)^{p-2}|\xi|\,dt \\ &\leq \frac{c_1}{p-1}(R+|\xi|)^{p-1}.\end{aligned}$$

A typical example of operator a satisfying $\mathrm{H}(a)_2$ is $a(x,\xi) = \theta(x)(R+|\xi|)^{p-2}\xi$ where $R \geq 0$ is a constant and $\theta \in C^1(\overline{\Omega},(0,+\infty))$.

For $R = 0$ and $\theta \equiv 1$, the resulting operator $\mathrm{div}\,a(x,\nabla u)$ is the p-Laplacian.

The previous examples (c) and (d) (under some restrictions) also satisfy $\mathrm{H}(a)_2$.

Theorem 2. *Assume* $\mathrm{H}(a)_2$ *and* $\mathrm{H}(f)$*. Let* u *be a weak solution of (1) or (2).*
(So we know that $u \in L^\infty(\Omega)$ by virtue of Moser iteration technique (Theorem 1)).
Then $u \in C^{1,\lambda}(\overline{\Omega})$ *for some* $\lambda \in (0,1)$*.*
Moreover, given $m \geq \|u\|_\infty$*, there are constants* $\lambda \in (0,1)$ *and* $M > 0$ *depending only on* m*,* c_0*,* c_1*,* R*,* α*,* Ω*,* p*, and* N*, such that*

$$u \in C^{1,\lambda}(\overline{\Omega}) \quad \textit{and} \quad \|u\|_{C^{1,\lambda}(\overline{\Omega})} \leq M.$$

Remark 3. If u is a weak solution of (1), then $u \in C^1(\overline{\Omega})$ (by Theorem 2),
and $u \in W_0^{1,p}(\Omega)$ (by definition).
Therefore, $u|_{\partial\Omega} = 0$, so u satisfies the boundary condition in problem (1).

Stronger hypotheses on the operator

$\mathrm{H}(a)$ **(i)** $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous, C^1 on $\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\})$, and $a(x, 0) = 0$ for all $x \in \overline{\Omega}$. Moreover, a is of the form

$$a(x, \xi) = \hat{a}(x, |\xi|)\xi \text{ for all } x \in \overline{\Omega}, \text{ all } \xi \in \mathbb{R}^N \setminus \{0\},$$

where $\hat{a} \in C^1(\overline{\Omega} \times (0, +\infty), (0, +\infty))$;

(ii) there is a constant $c_0 > 0$ such that, for all $x \in \overline{\Omega}$, $\xi, \eta \in \mathbb{R}^N$, $\xi \neq 0$, we have

$$(a'_\xi(x, \xi)\eta, \eta)_{\mathbb{R}^N} \geq c_0|\xi|^{p-2}\,|\eta|^2;$$

(iii) there is a constant $c_1 > 0$ such that, for all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$, $\xi \neq 0$, we have

$$\|a'_\xi(x, \xi)\| \leq c_1|\xi|^{p-2};$$

(iv) there is a constant $\alpha > 0$ such that, for all $x, y \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$, we have

$$|a(x, \xi) - a(y, \xi)| \leq c_1|x - y|^\alpha(1 + |\xi|)^{p-2}|\xi|.$$

Hypotheses $\mathrm{H}(a)$ are satisfied by the previous examples (a), (c) (for all $p \in (1, +\infty)$), and (d) with ($p \in (1, 2]$ and $c \in (0, 4p(p-1))$) or ($p > 2$ and $c \in (0, 2p + 2\sqrt{2p})$).

Moreover, the class of maps $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying $\mathrm{H}(a)$ is stable by addition and by multiplication by any map $\theta \in C^1(\overline{\Omega}, (0, +\infty))$.

Remark 4.

(a) $\mathrm{H}(a)$ is stronger than $\mathrm{H}(a)_2$ and so than $\mathrm{H}(a)_1$.

(b) Hypothesis $\mathrm{H}(a)$ (ii) implies that the operator a is strictly monotone, i.e., there is a constant $\tilde{c}_0 > 0$ such that, for $x \in \overline{\Omega}$, $\eta, \xi \in \mathbb{R}^N$,

$$(a(x,\xi) - a(x,\eta), \xi - \eta)_{\mathbb{R}^N} \geq \tilde{c}_0 |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2}.$$

Theorem 3. *Assume that* $\mathrm{H}(a)$ *holds. Then,* $V : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *defined by*

$$\langle V(u), v\rangle := \int_\Omega (a(x, \nabla u), \nabla v)_{\mathbb{R}^N}\, dx \quad \text{for all } u, v \in W^{1,p}(\Omega)$$

is an $(S)_+$*-map, that is, for every sequence* $\{u_n\}_{n\geq 1} \subset W^{1,p}(\Omega)$ *such that*

$$u_n \rightharpoonup u \ \text{ in } W^{1,p}(\Omega) \ \text{ and } \ \limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \leq 0$$

we have $u_n \to u$ *in* $W^{1,p}(\Omega)$.

Proof for p-Laplacian

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $1 < p < +\infty$ and consider the negative p-Laplacian $-\Delta_p : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by

$$\langle -\Delta_p(u), v\rangle = \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla v)_{\mathbb{R}^N}\, dx \quad \text{for all } u, v \in W^{1,p}(\Omega).$$

This map is an $(S)_+$-map. In particular, the negative p-Laplacian Dirichlet operator $(-\Delta_p, W_0^{1,p}(\Omega))$ is an $(S)_+$-map.

Indeed, we note that $-\Delta_p$ is monotone and continuous, hence it is maximal monotone.

So $-\Delta_p$ is generalized pseudomonotone.

Thus, if $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to\infty}\langle -\Delta_p(u_n), u_n - u\rangle \leq 0$, then $\|\nabla u_n\|_p^p = \langle -\Delta_p(u_n), u_n\rangle \to \langle -\Delta_p(u), u\rangle = \|\nabla u\|_p^p$.

Since $u_n \to u$ in $L^p(\Omega)$, we have that $\|u_n\| \to \|u\|$ (the Sobolev norm on $W^{1,p}(\Omega)$).

Because $W^{1,p}(\Omega)$ is uniformly convex and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$,

from Kadec–Klee property we conclude that $u_n \to u$ in $W^{1,p}(\Omega)$.

Let $\{u_n\}_{n\geq 1} \subset W^{1,p}(\Omega)$ be such that

$$(6) \qquad u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega) \text{ and } \limsup_{n\to\infty} \langle V(u_n), u_n - u\rangle \leq 0.$$

The first part of (6) yields $M_1 > 0$ such that

$$(7) \qquad \|\nabla u_n\|_p \leq M_1 \text{ for all } n \geq 1.$$

Note that $\langle V(u), u_n - u\rangle \to 0$ as $n \to \infty$.

Then, the monotonicity of a (see Remark 4 (b)) and the second part of (6) imply

$$\lim_{n\to\infty} \int_\Omega (a(x, \nabla u_n) - a(x, \nabla u), \nabla u_n - \nabla u)_{\mathbb{R}^N}\, dx = 0.$$

Hence

$$(8) \qquad w_n \to 0 \text{ in } L^1(\Omega)$$

where $w_n(x) := (a(x, \nabla u_n(x)) - a(x, \nabla u(x)), \nabla u_n(x) - \nabla u(x))_{\mathbb{R}^N} \geq 0$.

Claim 1: $\nabla u_n \to \nabla u$ a.e. in Ω.

We establish Claim 1 up to a subsequence of $\{\nabla u_n\}_{n\geq 1}$.

From (8), up to passing to a subsequence, we find $h \in L^1(\Omega)_+$ such that

$$w_n \to 0 \text{ a.e. in } \Omega \text{ and } 0 \leq w_n(x) \leq h(x) \text{ for a.a. } x \in \Omega, \text{ all } n \geq 1. \tag{9}$$

From (9), Remark 4 (b), there is a measurable subset $S \subset \Omega$ with $|S|_N = 0$ such that

$$\tilde{c}_0|\nabla u_n(x) - \nabla u(x)|^2(|\nabla u_n(x)| + |\nabla u(x)|)^{p-2} \leq h(x) \text{ for all } x \in \Omega \setminus S.$$

This implies that $\{\nabla u_n(x)\}_{n\geq 1}$ is bounded for all $x \in \Omega \setminus S$.

We claim that

$$\nabla u_n(x) \to \nabla u(x) \text{ for all } x \in \Omega \setminus S. \tag{10}$$

Fix $x \in \Omega \setminus S$. Since $\{\nabla u_n(x)\}_{n\geq 1}$ is bounded in $\mathbb{R}^N$, along a subsequence (depending on x) we get $\nabla u_{n_k}(x) \to \xi(x) \in \mathbb{R}^N$ as $k \to \infty$. The first part of (9) then yields

$$(a(x, \xi(x)) - a(x, \nabla u(x)), \xi(x) - \nabla u(x))_{\mathbb{R}^N} = 0.$$

The strict monotonicity of a (see Remark 4 (b)) yields $\nabla u(x) = \xi(x)$. Whence (10).

Claim 2: For every $\varepsilon > 0$, we can find $m > 0$ such that, for every $n \geq 1$,

$$\int_{\Omega_{n,m}} |\nabla u_n(x)|^p \, dx \leq \varepsilon, \quad \text{where } \Omega_{n,m} := \{x \in \Omega : \ |\nabla u_n(x)|^p > m\}.$$

First, we note that (7) and the definition of $\Omega_{n,m}$ yield, for all $m \in (0, +\infty)$,

$$(11) \qquad |\Omega_{n,m}|_N \leq \frac{1}{m^p} \int_{\Omega_{n,m}} |\nabla u_n(x)|^p \, dx \leq \frac{M_1^p}{m^p} \ \text{ for all } n \geq 1.$$

In view of (8), there is $n_0 \geq 1$ such that, for all $n \geq n_0$, we have

$$(12) \qquad \int_\Omega w_n(x) \, dx \leq \frac{c_0}{p-1} \, \frac{\varepsilon}{2} \, .$$

Proof: Claim 2 (continue)

We can find $\delta > 0$ such that, every measurable set $A \subset \Omega$ with $|A|_N \le \delta^p$ satisfies

$$(13) \qquad \int_A |\nabla u_n(x)|^p \, dx \le \varepsilon \ \text{ for all } n \in \{1, \dots n_0 - 1\},$$

and (letting $I_A(u) = \big(\int_A |\nabla u(x)|^p \, dx \big)^{\frac{1}{p}}$)

$$(14) \qquad I_A(u)^p + I_A(u)^{p-1} M_1 + I_A(u) M_1^{p-1} \le \frac{c_0}{c_1} \frac{\varepsilon}{2}\,.$$

Set $m = \frac{M_1}{\delta} > 0$. Thus we have $|\Omega_{n,m}|_N \le \delta^p$ (see (11)), so that the claimed relation holds for every $n \in \{1, \dots, n_0 - 1\}$ (by (13)).
For $n \ge n_0$, using Remark 2, (12), (7), (14), and Hölder's inequality, we compute

$$\int_{\Omega_{n,m}} |\nabla u_n(x)|^p \, dx \le \frac{p-1}{c_0} \int_{\Omega_{n,m}} (a(x, \nabla u_n), \nabla u_n)_{\mathbb{R}^N} \, dx$$

$$\le \frac{p-1}{c_0} \Big(\int_\Omega w_n(x) \, dx + \int_{\Omega_{n,m}} \big((a(x, \nabla u), \nabla u_n - \nabla u)_{\mathbb{R}^N} + (a(x, \nabla u_n), \nabla u)_{\mathbb{R}^N} \big) \, dx \Big)$$

$$\le \frac{\varepsilon}{2} + \frac{c_1}{c_0} \int_{\Omega_{n,m}} \big(|\nabla u|^{p-1} (|\nabla u_n| + |\nabla u|) + |\nabla u_n|^{p-1} |\nabla u| \big) \, dx$$

$$\le \frac{\varepsilon}{2} + \frac{c_1}{c_0} \left(I_{\Omega_{n,m}}(u)^p + I_{\Omega_{n,m}}(u)^{p-1} M_1 + I_{\Omega_{n,m}}(u) M_1^{p-1} \right) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ ($\Rightarrow$ Claim 2).

Proof: conclusion

The property shown in Claim 2 means that the family $\{|\nabla u_n|^p\}_{n\geq 1} \subset L^1(\Omega)$ is uniformly integrable.

Together with the fact that $|\nabla u_n(x)|^p \to |\nabla u(x)|^p$ for a.a. $x \in \Omega$ (see Claim 1), this property permits the use of Vitali's theorem which implies that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx = \int_\Omega |\nabla u|^p \, dx.$$

Since $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, we have $\|u_n\|_p \to \|u\|_p$ as $n \to \infty$.

Whence $\|u_n\| \to \|u\|$.

Since $W^{1,p}(\Omega)$ satisfies the Kadec–Klee property, the facts that $u_n \rightharpoonup u$ and $\|u_n\| \to \|u\|$ ensure that $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$.

This shows that V is an $(S)_+$-map.

We define $G : \overline{\Omega} \times \mathbb{R}^N \longrightarrow [0, +\infty)$ by

$$G(x,\xi) = \int_0^{|\xi|} \hat{a}(x,t)t\,dt \quad \text{for all } x \in \overline{\Omega}, \text{ all } \xi \in \mathbb{R}^N.$$

Lemma. (a) *For every $x \in \overline{\Omega}$, the map $\xi \mapsto G(x,\xi)$ is of class C^1 and we have*

$$G'_\xi(x,\xi) = a(x,\xi) \quad \text{for all } x \in \overline{\Omega}, \text{ all } \xi \in \mathbb{R}^N.$$

(b) *For every $x \in \overline{\Omega}$, the map $\xi \mapsto G(x,\xi)$ is convex.*

(c) *We have*

$$(a(x,\xi),\xi)_{\mathbb{R}^N} \geq G(x,\xi) \geq \frac{c_0}{p(p-1)}\,|\xi|^p \quad \text{and} \quad G(x,\xi) \leq \frac{c_1}{p(p-1)}\,|\xi|^p$$

for all $x \in \overline{\Omega}$, all $\xi \in \mathbb{R}^N$, with c_0, c_1 from $\mathrm{H}(a)$.

Proof of Lemma

(a) The chain rule guarantees that $G(x,\cdot)$ is differentiable at every $\xi \neq 0$ and

$$G'_\xi(x,\xi) = \hat{a}(x,|\xi|)|\xi|\,\frac{\xi}{|\xi|} = a(x,\xi).$$

It follows from the relation $\hat{a}(x,t)t = |a(x,t\frac{\xi}{|\xi|})|$ and the estimate for $|a(x,\xi)|$ pointed out in Remark 2 that $G(x,\cdot)$ is also differentiable at 0 with $G'_\xi(x,0) = 0 = a(x,0)$.

(b) Note that $G(x,\xi) = G_0(x,|\xi|)$ with $G_0(x,s) = \int_0^s \hat{a}(x,t)t\,dt$ for all $s \geq 0$.

It follows from the relation $\hat{a}(x,s)s = (a(x,s\xi),\xi)_{\mathbb{R}^N}$ for $|\xi| = 1$ and from the monotonicity of a (see Remark 4 (b)) that $s \mapsto \hat{a}(x,s)s$ is nondecreasing,

hence $G_0(x,\cdot)$ is convex. Therefore, $G(x,\cdot)$ is also convex.

(c) Using (a), we see that

$$G(x,\xi) = \int_0^1 (G'_\xi(x,t\xi),\xi)_{\mathbb{R}^N}\,dt = \int_0^1 (a(x,t\xi),\xi)_{\mathbb{R}^N}\,dt.$$

The conclusion follows from Remark 2 and the monotonicity of a (see Remark 4 (b)).

Variational setting

Hereafter, we assume $\mathrm{H}(a)$ and $\mathrm{H}(f)$.

We denote $F(x,s) = \int_0^s f(x,t)\,dt$.

The following proposition is a straightforward consequence of the Lemma.

Proposition. *The functional $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ defined by*

$$\varphi(u) = \int_\Omega G(x, \nabla u)\, dx - \int_\Omega F(x,u)\, dx \;\; \textit{for all } u \in W^{1,p}(\Omega)$$

is of class C^1, and we have

$$\langle \varphi'(u), v \rangle = \int_\Omega (a(x,\nabla u), \nabla v)_{\mathbb{R}^N}\, dx - \int_\Omega f(x,u(x))v(x)\, dx \;\; \textit{for all } u, v \in W^{1,p}(\Omega).$$

In particular, the critical points of φ coincide with the weak solutions of (2).

The critical points of $\varphi_0 := \varphi|_{W_0^{1,p}(\Omega)}$ coincide with the weak solutions of (1).

Theorem 4. *Assume* $\mathrm{H}(a)$ *and* $\mathrm{H}(f)$.

Let (X, ψ) *be any of the pairs* $(W^{1,p}(\Omega), \varphi)$ *or* $(W_0^{1,p}(\Omega), \varphi_0)$, *and let* $u_0 \in X$.

If u_0 *is a local minimizer of* ψ *with respect to the topology of* $C^1(\overline{\Omega})$,

i.e., there exists $\varepsilon > 0$ *such that*

$$\psi(u_0) \leq \psi(u_0 + h) \ \text{ for all } h \in X \cap C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq \varepsilon,$$

then u_0 *is a local minimizer of* ψ *with respect to the topology of* $W^{1,p}(\Omega)$,

i.e., there exists $\delta > 0$ *such that*

$$\psi(u_0) \leq \psi(u_0 + h) \ \text{ for all } h \in X \text{ with } \|\nabla h\|_p + \|h\|_p \leq \delta.$$

Proof: Claim 1

We start with pointing out a first consequence of the assumptions:

Claim 1: u_0 is a critical point of ψ and $u_0 \in C^1(\overline{\Omega})$.

Indeed, the assumption on u_0 implies that $\langle \psi'(u_0), h \rangle = 0$ for all $h \in X \cap C^1(\overline{\Omega})$.

Since $X \cap C^1(\overline{\Omega})$ is dense in X, we deduce $\psi'(u_0) = 0$,

i.e., u_0 is a critical point of ψ.

The fact that $u_0 \in C^1(\overline{\Omega})$ is then implied by the Proposition and Theorem 2 (Lieberman).

This shows Claim 1.

Proof by contradiction

Assume that u_0 is *not* a local $W^{1,p}(\Omega)$-minimizer of ψ.

Then, the continuity of the embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ (with $r \in (p, p^*)$ as in $\mathrm{H}(f)$) implies that, for every $\delta > 0$, we have

$$m_\delta := \inf\{\psi(u_0 + h) : \ h \in X, \ \|h\|_r \leq \delta\} < \psi(u_0).$$

Note that $m_\delta > -\infty$ (by $\mathrm{H}(f)$),

$\psi(u_0 + \cdot)$ is sequentially weakly l.s.c. on X and coercive on $\{h \in X : \ \|h\|_r \leq \delta\}$ (by the Lemma and $\mathrm{H}(f)$).

Thus, we can find $h_\delta \in X \setminus \{0\}$ such that

$$\|h_\delta\|_r \leq \delta \ \text{ and } \ \psi(u_0 + h_\delta) = m_\delta < \psi(u_0).$$

This and the definition of ψ (and using Lemma (c)) yield $M_1 > 0$ such that

$$\|\nabla h_\delta\|_p + \|h_\delta\|_p \leq M_1 \ \text{ for all } \delta \in (0, 1).$$

For the moment, we fix $\delta \in (0, 1)$ and study h_δ.

Proof: Claim 2

Claim 2: There is $\lambda_\delta \geq 0$ such that

$$-\operatorname{div} a(x, \nabla(u_0 + h_\delta)) = f(x, u_0 + h_\delta) - \lambda_\delta |h_\delta|^{r-2} h_\delta \quad \text{in } X^*.$$

Indeed, since $h_\delta \neq 0$, we have $\rho_\delta := \|h_\delta\|_r^r > 0$ and, by the choice of h_δ,

$$\psi(u_0 + h_\delta) = \inf\{\psi(u_0 + h) : \ h \in X, \ \|h\|_r^r = \rho_\delta\}.$$

Then, the Lagrange multiplier rule yields $\lambda_\delta \in \mathbb{R}$ with

$$\psi'(u_0 + h_\delta) = -\lambda_\delta |h_\delta|^{r-2} h_\delta \ \text{ in } X^*,$$

and we can see $\lambda_\delta \geq 0$ (by using h_δ as test function).

Claim 3: $h_\delta \in L^\infty(\Omega)$ and there is $M_2 > 0$ (independent of δ) such that $\|h_\delta\|_\infty \le M_2$.

By Claim 1, $-\operatorname{div} a(x, \nabla u_0) = f(x, u_0)$ in X^*. Subtracting from Claim 2, we obtain

$$-\operatorname{div} a_0(x, \nabla h_\delta) = f_0(x, h_\delta) - \lambda_\delta |h_\delta|^{p-2} h_\delta \ \text{ in } X^*,$$

with $a_0(x,\xi) = a(x, \nabla u_0(x) + \xi) - a(x, \nabla u_0(x))$, $f_0(x,s) = f(x, u_0(x) + s) - f(x, u_0(x))$.

By $\mathrm{H}(f)$ and $u_0 \in C^1(\overline{\Omega})$ (see Claim 1), we see that, for $x \in \Omega$, $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

$$\begin{aligned}
|f_0(x,s)| &\le c\big(2 + (\|u_0\|_\infty + |s|)^{r-1} + \|u_0\|_\infty^{r-1}\big) \le \tilde{c}(1 + |s|^{r-1}) \\
|a_0(x,\xi)| &\le \frac{c_1}{p-1}\big((\|\nabla u_0\|_\infty + |\xi|)^{p-1} + \|\nabla u_0\|_\infty^{p-1}\big) \le \tilde{c}_1(1 + |\xi|^{p-1}) \\
(a_0(x,\xi),\xi)_{\mathbb{R}^N} &\ge \tilde{c}_0(|\nabla u_0(x) + \xi| + |\nabla u_0(x)|)^{p-2}|\xi|^2 \ge \tilde{c}_0(R + |\xi|)^{p-2}|\xi|^2
\end{aligned}$$

for some constants $\tilde{c}, \tilde{c}_1 > 0$, for $\tilde{c}_0 > 0$ as before (see Remark 4 (b)),
and $R = 0$ (if $p \ge 2$) or $R = 2\|\nabla u_0\|_\infty$ (if $1 < p < 2$). (All constants independent of δ).

Proof: Claim 3 (continuing)

Exploiting the fact that $\lambda_\delta \geq 0$ (see Claim 2), we get

$$\int_\Omega (a_0(x, \nabla h_\delta), \nabla h)_{\mathbb{R}^N}\, dx \leq \int_\Omega f_0(x, h_\delta) h\, dx$$

for all h of the form $h = \min\{h_\delta^+, \lambda\}^\alpha$ or $h = -\min\{h_\delta^-, \lambda\}^\alpha$ with $\lambda > 0$ and $\alpha \geq 1$.

Applying Theorem 1, we obtain that $h_\delta \in L^\infty(\Omega)$ and $\|h_\delta\|_\infty \leq M_2$
for some $M_2 > 0$ independent of $\delta \in (0, 1)$.

This establishes Claim 3.

Proof: Claim 4

Claim 4: There is $M_3 > 0$ independent of δ such that $\lambda_\delta \|h_\delta\|_\infty^{r-1} \le M_3$.

Recall that $h_\delta \neq 0$ and set $\rho = \frac{1}{2}\|h_\delta\|_\infty > 0$.

Thus $\{h_\delta > \rho\} \neq \emptyset$ or $\{h_\delta < -\rho\} \neq \emptyset$. Say $\{h_\delta > \rho\} \neq \emptyset$ (the other case is similar).

Acting with the test function $(h_\delta - \rho)^+ \in X$, we get

$$\begin{aligned} 0 &\le \int_{\{h_\delta > \rho\}} (a_0(x, \nabla h_\delta), \nabla h_\delta)_{\mathbb{R}^N}\, dx \quad \text{(property of } a_0 \text{ seen above)} \\ &= \int_{\{h_\delta > \rho\}} f_0(x, h_\delta(x))(h_\delta(x) - \rho)\, dx - \lambda_\delta \int_{\{h_\delta > \rho\}} h_\delta(x)^{r-1}(h_\delta(x) - \rho)\, dx \\ &\le (\tilde{c}(1 + \rho^{1-r}) - \lambda_\delta) \int_{\{h_\delta > \rho\}} h_\delta(x)^{r-1}(h_\delta(x) - \rho)\, dx \quad \text{(property of } f_0\text{)}, \end{aligned}$$

whence $\lambda_\delta \le \tilde{c}(1 + \rho^{1-r})$ (since $\{h_\delta > \rho\} \neq \emptyset$). Using Claim 3, we deduce

$$\lambda_\delta \|h_\delta\|_\infty^{r-1} \le \tilde{c}(1 + 2^{r-1}\|h_\delta\|_\infty^{1-r})\|h_\delta\|_\infty^{r-1} \le \tilde{c}(M_2^{r-1} + 2^{r-1}) =: M_3,$$

whence Claim 4.

Proof concluded

By $\mathrm{H}(f)$ and Claims 1, 3, 4, we find $M_4 > 0$ independent of δ such that, for $x \in \Omega$,

$$-M_4 \leq \tilde{f}(x) := f(x, u_0(x) + h_\delta(x)) - \lambda_\delta |h_\delta(x)|^{r-2} h_\delta(x) \leq M_4$$

while Claim 2 reads as

$$-\mathrm{div}\, a(x, \nabla(u_0 + h_\delta)) = \tilde{f}(x) \text{ in } X^*.$$

Claim 3 yields $\|u_0 + h_\delta\|_\infty \leq \|u_0\|_\infty + M_2$ (independent of δ).

Theorem 2 (Lieberman) provides $\theta \in (0,1)$ and $M_5 > 0$ both independent of δ such that $u_0 + h_\delta \in C^{1,\theta}(\overline{\Omega})$ and $\|u_0 + h_\delta\|_{C^{1,\theta}(\overline{\Omega})} \leq M_5$ for all $\delta \in (0,1)$.

The compactness of the embedding $C^{1,\theta}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ yields a sequence $\{\delta_n\}_{n \geq 1}$ with $\delta_n \to 0$ such that

$$\{h_{\delta_n}\} \text{ converges in } C^1(\overline{\Omega}), \quad \text{in fact } \lim_{n \to \infty} \|h_{\delta_n}\|_{C^1(\overline{\Omega})} = 0 \quad (\text{since } \|h_{\delta_n}\|_r \leq \delta_n).$$

Since $\psi(u_0 + h_{\delta_n}) < \psi(u_0)$, this contradicts the assumption that u_0 is a local $C^1(\overline{\Omega})$-minimizer of ψ. The proof of the theorem is complete.

Theorem 4 in the Neumann case is due to Motreanu–Papageorgiou [7] and was extended in Miyajima–Motreanu–Tanaka [4].

The unified approach for Dirichlet and Neumann cases presented here is taken from the book of Motreanu–Motreanu–Papageorgiou [6].

The first such result was proved for $H_0^1(\Omega)$ with $a(x,\xi)=\xi$ by Brezis–Nirenberg [1].

It was extended to the space $W_0^{1,p}(\Omega)$ with $a(x,\xi)=|\xi|^{p-2}\xi$
by García Azorero–Manfredi–Peral Alonso [2] and Guo–Zhang (for $p\geq 2$).

For the space $W^{1,p}(\Omega)$ with $a(x,\xi)=|\xi|^{p-2}\xi$, it was proved by Barletta–Papageorgiou (for $2\leq p<+\infty$) and by Motreanu–Motreanu–Papageorgiou [5] (for $1<p<+\infty$).

Note that the argument of proof in [7], [4], [6] is different and more direct than the proofs in the aforementioned works.

Multiplicity result: Hypotheses and example

$\mathrm{H}(f)'$ **(i)** The map $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f(x,0) = 0$ a.e. in Ω and there are $r \in (p, p^*)$ and $c > 0$ such that

$$|f(x,s)| \leq c(1 + s^{r-1}) \text{ for a.a. } x \in \Omega, \text{ all } s \in [0, +\infty);$$

(ii) $\displaystyle\lim_{s \to +\infty} \frac{F(x,s)}{s^p} = 0$ and $\displaystyle\lim_{s \to +\infty} F(x,s) = -\infty$ uniformly for a.a. $x \in \Omega$

(where $F(x,s) = \int_0^s f(x,t)\, dt$);

(iii) there is a constant $c_+ > 0$ such that $\displaystyle\int_\Omega F(x, c_+) > 0$;

(iv) there is $\delta > 0$ such that $f(x,s) \leq 0$ for a.a. $x \in \Omega$, all $s \in [0, \delta]$.

Example. The following function f fulfills the hypotheses $\mathrm{H}(f)'$ (for simplicity we drop the x-dependence):

$$f(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ -s^{\tau-1} + 2s^{\gamma-1} & \text{if } s \in (0,1], \\ 2s^{\theta-1} - s^{q-1} & \text{if } s > 1, \end{cases}$$

where $1 < \tau < \gamma < 2\tau < +\infty$ and $1 < \theta < q < p$.

Theorem 5. *Assume that* $\mathrm{H}(a)$ *and* $\mathrm{H}(f)'$ *hold. Then, problem (2) admits at least two nontrivial solutions* $u_0, v_0 \in C^1(\overline{\Omega})$ *with* $0 \le v_0 \le u_0$ *in* Ω*.*

Proof. We deal with the functional $\varphi_+ \in C^1(W^{1,p}(\Omega), \mathbb{R})$ defined by

$$\varphi_+(u) = \int_\Omega G(x, \nabla u)\, dx + \frac{1}{p} \|u^-\|_p^p - \int_\Omega F(x, u^+)\, dx \quad \text{for all } u \in W^{1,p}(\Omega).$$

The proof splits into several steps.

Step 1: φ_+ is sequentially weakly lower semicontinuous, bounded below, and coercive.

The sequential weak lower semicontinuity of φ_+ is a consequence of $\mathrm{H}(f)'$ (i) and of the fact that $G(x, \cdot)$ is continuous and convex (see the Lemma).

Proof of Step 1: coercivity of φ_+

Arguing by contradiction, assume that we can find $\{u_n\}_{n\geq 1} \subset W^{1,p}(\Omega)$ and $M_1 > 0$ such that

$$(15) \qquad \|u_n\| \to +\infty \text{ as } n \to \infty \text{ and } \varphi_+(u_n) \leq M_1 \text{ for all } n \geq 1.$$

Using Lemma (c), the second relation in (15) implies

$$(16) \qquad \frac{c_0}{p(p-1)} \|\nabla u_n\|_p^p + \frac{1}{p}\|u_n^-\|_p^p - \int_\Omega F(x, u_n^+)\, dx \leq M_1 \text{ for all } n \geq 1.$$

From the first part of (15), (16), and the growth condition in $\mathrm{H}(f)'$ (i), it follows that $\|u_n^+\| \to +\infty$ as $n \to \infty$.

Thus, there is $y \in W^{1,p}(\Omega)$ such that, along a relabeled subsequence, we have

$$(17) \qquad y_n := \frac{u_n^+}{\|u_n^+\|} \rightharpoonup y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^\theta(\Omega) \text{ for each } \theta \in (1, p^*).$$

We claim that $y \equiv \left(\frac{1}{|\Omega|_N}\right)^{\frac{1}{p}}$.

From (16), we have

$$(18) \qquad \frac{c_0}{p(p-1)} \|\nabla y_n\|_p^p - \int_\Omega \frac{F(x,u_n^+)}{\|u_n^+\|^p}\,dx \leq \frac{M_1}{\|u_n^+\|^p} \quad \text{for all } n \geq 1.$$

Note that

$$(19) \qquad \lim_{n\to\infty} \int_\Omega \frac{F(x,u_n^+)}{\|u_n^+\|^p}\,dx = 0.$$

Indeed, invoking $\mathrm{H}(f)'$ (ii), (iii), we can find $c_\varepsilon > 0$ such that

$$|F(x,s)| \leq \frac{\varepsilon}{2}|s|^p + c_\varepsilon \quad \text{for a.a. } x \in \Omega, \text{ all } s \in \mathbb{R},$$

which yields, for $n \geq 1$ large enough (recall that $\|u_n^+\| \to +\infty$ as $n \to \infty$),

$$\int_\Omega \frac{|F(x,u_n^+)|}{\|u_n^+\|^p}\,dt \leq \frac{\varepsilon}{2} + \frac{c_\varepsilon|\Omega|_N}{\|u_n^+\|^p} \leq \varepsilon.$$

This shows (19). Passing to $\limsup$ in (18) as $n \to \infty$ and using (17), (19), we have

$$(20) \qquad \limsup_{n\to\infty} \|\nabla y_n\|_p^p \leq 0 \leq \|\nabla y\|_p^p\,.$$

By (17) we have $\|\nabla y\|_p^p \leq \liminf_{n\to\infty} \|\nabla y_n\|_p^p$, hence $\nabla y_n \to \nabla y = 0$ in $L^p(\Omega)$ (by (20)). So

$$y_n \to y \ \text{ in } W^{1,p}(\Omega) \ \text{(see (17)), and so } \|y\| = 1,\ y \geq 0.$$

Since $\nabla y = 0$, we have in fact $y \equiv \left(\frac{1}{|\Omega|_N}\right)^{\frac{1}{p}}$.

Proof of Step 1: conclusion

Thus

$$y_n = \frac{u_n^+}{\|u_n^+\|} \longrightarrow y \equiv \left(\frac{1}{|\Omega|_N}\right)^{\frac{1}{p}} \text{ in } L^p(\Omega).$$

So up to considering a subsequence, we may assume that $u_n^+(x) \longrightarrow +\infty$ for a.a. $x \in \Omega$.

Combining this with $\mathrm{H}(f)'$ (i), (ii), (16), and Fatou's lemma, we get

$$M_1 \geq \varphi_+(u_n) \geq -\int_\Omega F(x, u_n^+)\,dx \longrightarrow +\infty \text{ as } n \longrightarrow \infty,$$

a contradiction. Thereby φ_+ is coercive.

The coercivity of φ_+ and $(x, s) \longmapsto F(x, s^+)$ is bounded on bounded sets (see $\mathrm{H}(f)'$ (i)) imply that φ_+ is bounded below.

Step 1 is complete.

Step 2: φ_+ admits a global minimizer $u_0 \in W^{1,p}(\Omega)$. Moreover, we have $u_0 \in C^1(\overline{\Omega})$, $u_0 \geq 0$ in Ω, $u_0 \neq 0$, and u_0 is a solution of problem (2).

In view of Step 1, there exists $u_0 \in W^{1,p}(\Omega)$ which is a global minimizer of φ_+.

By $\mathrm{H}(f)'$ (iii), we have

$$\varphi_+(u_0) \leq \varphi_+(c_+) = -\int_\Omega F(x, c_+)\,dx < 0 = \varphi_+(0).$$

This ensures that $u_0 \neq 0$.

The fact that u_0 is a critical point of φ_+ yields

$$V(u_0) - (u_0^-)^{p-1} = f(x, u_0^+) \text{ in } W^{1,p}(\Omega)^*. \tag{21}$$

Acting on (21) with the test function $-u_0^-$, we obtain

$$\|u_0^-\|_p^p \leq \int_{\{u_0<0\}} (a(x, \nabla u_0), \nabla u_0)_{\mathbb{R}^N}\,dx + \|u_0^-\|_p^p = 0$$

hence $u_0^- = 0$, so $u_0 \geq 0$ a.e. in Ω. Thus (21) reads as

$$V(u_0) = f(x, u_0) \text{ in } W^{1,p}(\Omega)^*,$$

i.e., u_0 is a weak solution of problem (2).
Theorem 2 yields $u_0 \in C^1(\overline{\Omega})$. Therefore, u_0 satisfies all the claimed properties.

Proof: Step 3

In order to look for a further nonnegative solution for problem (2),
we define $\hat{\varphi}_+ \in C^1(W^{1,p}(\Omega), \mathbb{R})$ by

$$\hat{\varphi}_+(u) = \int_\Omega G(x, \nabla u)\, dx + \frac{1}{p}\, \|u^-\|_p^p + \frac{1}{p}\, \|(u - u_0)^+\|_p^p - \int_\Omega \hat{F}_+(x, u(x))\, dx$$

for all $u \in W^{1,p}(\Omega)$, with $\hat{F}_+(x,s) = \int_0^s f(x, \hat{\tau}_+(x,t))\, dt$, where

$$\hat{\tau}_+(x,s) = \begin{cases} 0 & \text{if } s \leq 0, \\ s & \text{if } 0 < s < u_0(x), \\ u_0(x) & \text{if } s \geq u_0(x), \end{cases}$$

for all $(x,s) \in \Omega \times \mathbb{R}$.

Step 3: If u is a critical point of $\hat{\varphi}_+$, then u is a solution of (2), $u \in C^1(\overline{\Omega})$, and we have $0 \leq u(x) \leq u_0(x)$ for all $x \in \Omega$.

Let u be a critical point of $\hat{\varphi}_+$. Then

$$(22) \qquad V(u) - (u^-)^{p-1} + ((u-u_0)^+)^{p-1} = f(x, \hat{\tau}_+(x, u(x))) \text{ in } W^{1,p}(\Omega)^*.$$

Acting on (22) with the test function $-u^-$, we obtain (as in Step 2) $u^- = 0$ a.e. in Ω.
Acting on (22) with the test function $(u-u_0)^+$, we have

$$\int_\Omega (a(x, \nabla u), \nabla(u-u_0)^+)_{\mathbb{R}^N}\, dx + \|(u-u_0)^+\|_p^p = \int_\Omega f(x, u_0(x))(u-u_0)^+\, dx.$$

The fact that $u_0 \geq 0$ is a critical point of φ_+ yields

$$-\int_\Omega (a(x, \nabla u_0), \nabla(u-u_0)^+)_{\mathbb{R}^N}\, dx = -\int_\Omega f(x, u_0(x))(u-u_0)^+\, dx.$$

Adding these inequalities and using the monotonicity of a (see Remark 4 (b)), we obtain

$$\|(u-u_0)^+\|_p^p \leq \int_{\{u>u_0\}} (a(x, \nabla u) - a(x, \nabla u_0), \nabla(u-u_0))_{\mathbb{R}^N}\, dx + \|(u-u_0)^+\|_p^p = 0,$$

whence $(u-u_0)^+ = 0$. Thus $0 \leq u \leq u_0$ a.e. in Ω. So (22) becomes

$$V(u) = f(x, u(x)) \text{ in } W^{1,p}(\Omega)^*,$$

so u is a solution of (2). By Theorem 2, we have $u \in C^1(\overline{\Omega})$,

and the relation $0 \leq u \leq u_0$ holds everywhere in Ω.

Proof: Step 4 and coercivity of $\hat{\varphi}_+$

Step 4: The functional $\hat{\varphi}_+$ satisfies the Palais–Smale condition.

We first check that $\hat{\varphi}_+$ is coercive.

Since $u_0 \in C^1(\overline{\Omega})$ and using $\mathrm{H}(f)'$ (i), we find a constant $\hat{c} > 0$ such that

$$\left| \int_\Omega \hat{F}_+(x,u)\,dx \right| \leq \hat{c}\|u\|_p \quad \text{for all } u \in W^{1,p}(\Omega).$$

Moreover, for every $u \in W^{1,p}(\Omega)$, we see that

$$\begin{aligned} \|(u-u_0)^+\|_p^p &= \|(u^+-u_0)^+\|_p^p = \|u^+-u_0\|_p^p - \int_{\{u^+\leq u_0\}} (u_0-u^+)^p\,dx \\ &\geq \frac{1}{2^{p-1}}\|u^+\|_p^p - \|u_0\|_p^P - \int_{\{u^+\leq u_0\}} u_0(x)^p\,dx \\ &\geq \frac{1}{2^{p-1}}\|u^+\|_p^p - 2\|u_0\|_p^p\,. \end{aligned}$$

From Lemma (c) and the above inequalities, we derive

$$\hat{\varphi}_+(u) \geq \frac{c_0}{p(p-1)}\|\nabla u\|_p^p + \frac{1}{p}\|u^-\|_p^p + \frac{1}{p2^{p-1}}\|u^+\|_p^p - \frac{2}{p}\|u_0\|_p^p - \hat{c}\|u\|_p$$

for all $u \in W^{1,p}(\Omega)$. Therefore $\hat{\varphi}_+$ is coercive.

Proof: Proof of Step 4

Let $\{u_n\}_{n\geq 1} \subset W^{1,p}(\Omega)$ be a sequence such that

(23) $\{\hat{\varphi}_+(u_n)\}_{n\geq 1}$ is bounded and $\hat{\varphi}_+'(u_n) \to 0$ in $W^{1,p}(\Omega)^*$ as $n \to \infty$.

By the first part of (23) and the coercivity of $\hat{\varphi}_+$, $\{u_n\}_{n\geq 1}$ is bounded in $W^{1,p}(\Omega)$.

So, there is $u \in W^{1,p}(\Omega)$ such that, along a relabeled subsequence $\{u_n\}_{n\geq 1}$, we have

(24) $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $u_n \to u$ in $L^\theta(\Omega)$ for each $\theta \in [1, p^*)$.

The second part of (23) yields

$$\int_\Omega (a(x, \nabla u_n), u_n - u)_{\mathbb{R}^N}\, dx - \int_\Omega \hat{g}_+(x, u_n(x))(u_n - u)\, dx \to 0 \text{ as } n \to \infty,$$

where $\hat{g}_+(x,s) = (s^-)^{p-1} - ((s - u_0(x))^+)^{p-1} + f(x, \hat{\tau}_+(x,s))$.

The growth condition in $\mathrm{H}(f)'$ (i) and the second part of (24) ensure that

$$\int_\Omega \hat{g}_+(x, u_n(x))(u_n(x) - u(x))\, dx \to 0 \text{ as } n \to \infty.$$

Whence $\lim\limits_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0$.

Since V is an $(S)_+$-map (by Theorem 3), we get that $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$.

This shows that $\hat{\varphi}_+$ satisfies the Palais–Smale condition.

Step 5: $\hat{\varphi}_+$ admits a critical point $v_0 \in W^{1,p}(\Omega)$ different from $0, u_0$.

Note that

$$(25) \qquad \hat{\varphi}_+(u_0) = \varphi_+(u_0) < \varphi_+(0) = \hat{\varphi}_+(0) = 0.$$

We claim that

$$(26) \qquad 0 \text{ is a local minimizer of } \hat{\varphi}_+.$$

Using $\mathrm{H}(f)'$ (iv) and the fact that $0 \leq \hat{\tau}_+(x,t) \leq |t|$ for a.a. $x \in \Omega$, all $t \in \mathbb{R}$, we have

$$(27) \qquad \hat{F}_+(x,s) := \int_0^s f(x, \hat{\tau}_+(x,t))\,dt \leq 0 \text{ for a.a. } x \in \Omega, \text{ all } s \in [-\delta, \delta].$$

Let $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1(\overline{\Omega})} \leq \delta$. By (27), we have $\hat{F}_+(x, u(x)) \leq 0$ for a.a. $x \in \Omega$, so

$$\hat{\varphi}_+(u) \geq -\int_\Omega \hat{F}_+(x, u(x))\,dx \geq 0.$$

This shows that 0 is a local minimizer of $\hat{\varphi}_+$ for the topology of $C^1(\overline{\Omega})$.

By Theorem 4, we infer that 0 is a local minimizer of $\hat{\varphi}_+$ for the topology of $W^{1,p}(\Omega)$.
Hence, we obtain (26).
We may assume that 0 is a strict local minimizer of $\hat{\varphi}_+$
(otherwise, any neighborhood of 0 in $W^{1,p}(\Omega)$ contains another critical point of $\hat{\varphi}$).

This fact, (25) and Step 4, by the mountain pass theorem, yield a critical point v_0 of $\hat{\varphi}_+$ different from 0 and u_0. This completes Step 5.
The theorem now follows by combining Steps 2, 3, and 5.

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