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Nonlinear Inclusions and Hemivariational Inequalities with Applications to Contact Mechanics. Part II: Evolution Problems

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Outline of the talk

- Parabolic hemivariational inequality
  - Example: the semipermeability problem
  - Variational formulation and existence results
- Hyperbolic hemivariational inequality
  - Detailed problem: dynamic viscoelastic contact
  - Variational formulation and existence results
  - Subdifferential boundary conditions in mechanics
- Optimal control problems for hemivariational inequalities
  - Examples for stationary and evolution hemivariational inequalities
- Recent advances in the theory of hemivariational inequalities

Based on some recent results from the following recent monograph.

S. Migorski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Advances in Mechanics and Mathematics, vol. **26**, Springer, New York, 2013, pages: 285.

The Clarke subdifferential

Given a locally Lipschitz function $h\colon E \to \mathbb{R}$, where E is a Banach space, we define (Clarke (1983)):

- **the generalized directional derivative** of h at $x \in E$ in the direction $v \in E$ by

$$h^0(x;v) = \limsup_{y\to x,\ t\downarrow 0} \frac{h(y+tv)-h(y)}{t}.$$

- **the generalized gradient** of h at x by $\partial h(x)$, is a subset of a dual space E^* given by

$$\partial h(x) = \{\zeta \in E^* \mid h^0(x;v) \geq \langle \zeta, v\rangle_{E^*\times E} \text{ for all } v \in E\}.$$

The locally Lipschitz function h is called **regular** (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $h'(x;v)$ exists and satisfies $h^0(x;v) = h'(x;v)$ for all $v \in E$.

Parabolic hemivariational inequalities

Motivation: Examples which come from the nonconvex problems for semipermeable media and lead to a scalar time dependent hemivariational inequalities. The nonmonotone semipermeability conditions are realized by various types of membranes, natural and artificial ones. These conditions arise in electrostatics, hydraulics and in the description of the flow of Bingham's fluids.

Consider the following parabolic hemivariational inequality:

$$\begin{cases} \text{find } u \in L^2(0,T;V) \text{ such that } u' \in L^2(0,T;V^*) \\ \langle u' + Au, v\rangle_{V^*\times V} + \int_\Omega j^0(t,u;v)\,dx \geq \langle f(t), v\rangle_{V^*\times V} \end{cases}$$

for a.e. $t \in (0,T)$ and all $v \in V$.

Results for parabolic hemivariational inequalities:

- monotone semipermeability relations leading to variational inequalities: Duvaut and Lions (1976)
- nonmonotone relations leading to hemivariational inequalities and one-dimensional superpotentials:
  - Panagiotopoulos (1993), Miettinen (1996): regularization technique with Galerkin method
  - Carl (1996): combined the method of upper and lower solutions, theory of pseudomonotone operators with truncation and penalization
  - Miettinen and Panagiotopoulos (1999): under assumption that the Nemitsky operator is potential
  - Haslinger, Miettinen and Panagiotopoulos (1999): nonmonotone potentials on the boundary and approximation schemes
- multidimensional nonconvex superpotentials: Liu (1999): for operators of class (S_+), Migorski (2000,2001): for pseudomonotone multivalued operators, Migorski and Ochal (2000): a regularized approximating method, Carl (2001): extremal solutions, Ochal (2001), Carl and Gilbert (2002); Motreanu, Sofonea, Migorski et al.

Example

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set and $0 < T < +\infty$. Consider the time-dependent heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f \text{ in } \Omega \times (0,T).$$

Suppose that $f = f_1 + f_2$ with f_2 prescribed and

$$-f_1(x,t) \in \partial j(x, u(x,t)) \text{ a.e. in } \Omega \times (0,T).$$

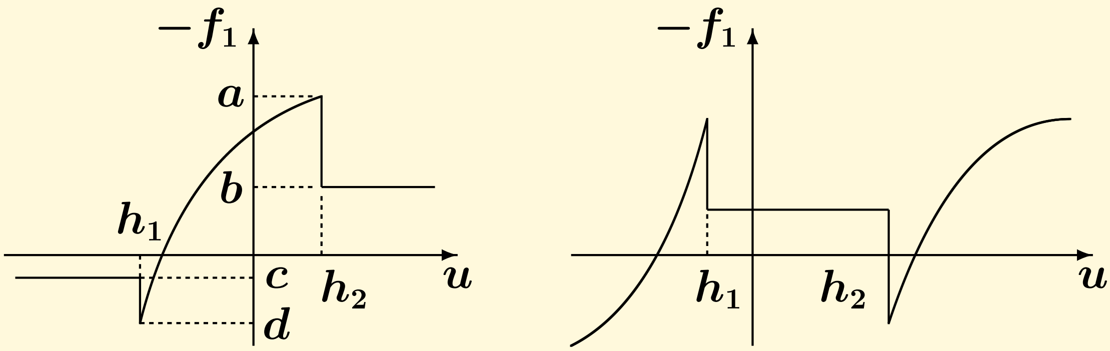
On the boundary $\partial\Omega$ the temperature u satisfies the Dirichlet (homogeneous, for simplicity) boundary condition

$$u = 0 \text{ on } \partial\Omega \times (0,T)$$

and moreover, at $t = 0$ the temperature is given

$$u(x,0) = u_0(x) \text{ in } \Omega.$$

The function $j\colon \Omega \times \mathbb{R} \to \mathbb{R}$, $j = j(x,\xi)$ is assumed to be measurable in x and locally Lipschitz in ξ.



(a) **(b)**

Semipermeability relations

Figure (a): behavior of a semipermeable membrane of finite thickness occupying the set Ω

Figure (b): behavior of temperature controller producing in Ω heat in order to regulate the temperature in the interior of Ω. The graph corresponds to a temperature control problem when we regulate the temperature to deviate as little as possible from the interval $[h_1, h_2]$

Taking into account the definition of generalized subdifferential, the weak formulation of the evolution problem for the heat equation is the following hemivariational inequality:

find $u\colon \Omega \times (0,T) \longrightarrow \mathbb{R}$ such that

$$\begin{cases} \langle u'(t), v\rangle_{V^*\times V} + \langle \nabla u(t), \nabla v\rangle_{V^*\times V} + \\ + \int_\Omega j^0(x, u(x,t); v)\, dx \geq \langle f_2(t), v\rangle_{V^*\times V} \\ \qquad \text{for a.e. } t \in (0,T) \text{ and all } v \in V = H_0^1(\Omega) \\ u(0) = u_0. \end{cases}$$

Hyperbolic hemivariational inequalities

Motivation: Mathematical Theory of Contact Mechanics deals with mathematical modelling and analysis of some of the phenomena that take place when a deformable body comes into contact. Two branches: quasistatic contact and dynamic contact. Complex phenomena: include friction, wear, adhesion, thermal effects, and material damage among others. There is a need to provide new mathematical results on modeling of various processes with nonconvex superpotentials.

Results on contact problems:
Signorini (1933), Fichera (1964), Duvaut and Lions (1976)
Nečas, Jarušek and Haslinger (1980)

books: Han and Sofonea (2002), Shillor, Sofonea and Telega (2004), Panagiotopoulos (1985,1993), Motreanu, Carl: theory of hemivariational inequalities, Migorski, Ochal and Sofonea (2013)

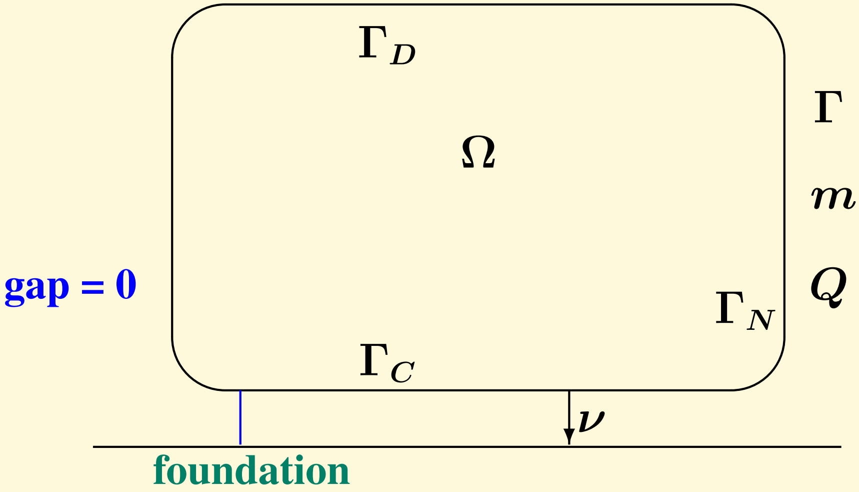
Results for hyperbolic hemivariational inequalities:

- one-dimensional superpotentials in hemivariational inequalities:
  - Panagiotopoulos (1995): problems with impacts
  - Gasiński and Smołka (2002): wave type hemivariational inequality
  - Panagiotopoulos and Pop (1999): hyperbolic variational-hemivariational inequlaities for linear operators via regularization method
  - Goeleven, Miettinen and Panagiotopoulos (1999): nonlinear damping operator
  - Haslinger, Miettinen and Panagiotopoulos (1999): approximation schemes with no convergence result
  - Park and Kim (2003): existence and asymptotic behavior
- nonmonotone relations coming form multidimensional superpotentials: Ochal (2001): surjectivity method for pseudomonotone operators, Sofonea, Motreanu, Migorski et al.

Hyperbolic hemivariational inequality

We formulate mathematical models describing dynamic viscoelastic contact problems with the Kelvin-Voigt constitutive law and subdifferential boundary conditions. We treat evolution hemivariational inequalities which are weak formulations of contact problems.

Physical setting



$\Gamma = \partial\Omega$

$m(\Gamma_D) > 0$

$Q = \Omega \times (0, T)$

$\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary occupied by a deformable viscoelastic body

$\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$ mutually disjoint parts

Γ_C the potential contact surface

We suppose that the body is clamped on Γ_D, so the displacement field vanishes there. Volume forces of density f_1 act in Ω and surface tractions of density f_2 are applied on Γ_N.

Problem formulation

Let $u\colon Q \longrightarrow \mathbb{R}^d$ be the displacement vector, $\sigma\colon Q \longrightarrow \mathcal{S}_d$ the stress tensor and $\varepsilon(u) = (\varepsilon_{ij}(u))$ the linearized strain tensor, where $i, j = 1, \ldots, d$. We employ the Kelvin-Voigt viscoelastic constitutive relation

$$\sigma(u, u') = \mathcal{C}(\varepsilon(u')) + \mathcal{G}(\varepsilon(u)),$$

where $\mathcal{C}$ and $\mathcal{G}$ are given constitutive functions. The contact problem can be stated as follows

$$\begin{cases} u''(t) - \operatorname{div} \sigma(u(t), u'(t)) = f_1(t) & \text{in } Q \\ \sigma(u, u') = \mathcal{C}(\varepsilon(u')) + \mathcal{G}(\varepsilon(u)) & \text{in } Q \\ u = 0 & \text{on } \Gamma_D \times (0, T) \\ \sigma\nu = f_2 & \text{on } \Gamma_N \times (0, T) \\ -\sigma_\nu(t) \in \partial j_\nu(x, t, u_\nu(t)) & \text{on } \Gamma_C \times (0, T) \\ -\sigma_\tau(t) \in \partial j_\tau(x, t, u'_\tau(t)) & \text{on } \Gamma_C \times (0, T) \\ u(0) = u_0, \ \ u'(0) = u_1 & \text{in } \Omega. \end{cases}$$

Hypotheses

$\underline{H(\mathcal{C})}$: The viscosity operator $\mathcal{C}: Q \times \mathcal{S}_d \longrightarrow \mathcal{S}_d$ satisfies the Carathéodory condition (i.e. $\mathcal{C}(\cdot, \cdot, \varepsilon)$ is measurable on Q for all $\varepsilon \in S_d$ and $\mathcal{C}(x, t, \cdot)$ is continuous on $\mathcal{S}_d$ for a.e. $(x, t) \in Q$) and

(i) $\|\mathcal{C}(x, t, \varepsilon)\|_{\mathcal{S}_d} \leq c_1 (b(x, t) + \|\varepsilon\|_{\mathcal{S}_d})$ for all $\varepsilon \in \mathcal{S}_d$ and a.e. $(x, t) \in Q$ with $b \in L^2(Q)$, $b \geq 0$ and $c_1 > 0$;

(ii) $(\mathcal{C}(x, t, \varepsilon_1) - \mathcal{C}(x, t, \varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geq 0$ for all $\varepsilon_1, \varepsilon_2 \in \mathcal{S}_d$ and a.e. $(x, t) \in Q$;

(iii) $\mathcal{C}(x, t, \varepsilon) : \varepsilon \geq c_2 \|\varepsilon\|^2_{\mathcal{S}_d}$ for all $\varepsilon \in \mathcal{S}_d$ and a.e. $(x, t) \in Q$ with $c_2 > 0$.

$\underline{H(\mathcal{G})}$: The elasticity operator $\mathcal{G}: \Omega \times \mathcal{S}_d \longrightarrow \mathcal{S}_d$ is of the form $\mathcal{G}(x, \varepsilon) = \mathbb{E}(x)\varepsilon$ (the Hooke law) with a symmetric and positive elasticity tensor $\mathbb{E} \in L^\infty(\Omega)$, i.e. $\mathbb{E} = (g_{ijkl})$, $i, j, k, l = 1, \ldots, d$ with $g_{ijkl} = g_{jikl} = g_{lkij}$ and $g_{ijkl}(x)\chi_{ij}\chi_{kl} \geq c_3 \chi_{ij}\chi_{ij}$ for all symmetric tensors $\chi = \{\chi_{ij}\}$ and for a.e. $x \in \Omega$ with $c_3 > 0$.

Hypotheses

$\underline{H(f)}$: $f_1 \in L^2(0,T;H)$, $f_2 \in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d))$, $u_0 \in V$, $u_1 \in H$.

$\underline{H(j_\nu)}$: $j_\nu \colon \Gamma_C \times (0,T) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that

(i) $j_\nu(\cdot,\cdot,\xi)$ is measurable for $\xi \in \mathbb{R}$, $j_\nu(\cdot,\cdot,0) \in L^1(\Gamma_C \times (0,T))$;

(ii) $j_\nu(x,t,\cdot)$ is locally Lipschitz for all $(x,t) \in \Gamma_C \times (0,T)$;

(iii) $|\partial j_\nu(x,t,\xi)| \leq c_\nu\,(1 + |\xi|)$ for all (x,t,ξ) with $c_\nu > 0$;

$\underline{H(j_\tau)}$: $j_\tau \colon \Gamma_C \times (0,T) \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is a function such that

(i) $j_\tau(\cdot,\cdot,\eta)$ is measurable, $j_\tau(\cdot,\cdot,0) \in L^1(\Gamma_C \times (0,T))$;

(ii) $j_\tau(x,t,\cdot)$ is locally Lipschitz for all $(x,t) \in \Gamma_C \times (0,T)$;

(iii) $\|\partial j_\tau(x,t,\eta)\|_{\mathbb{R}^d} \leq c_\tau\,(1 + \|\eta\|_{\mathbb{R}^d})$ for all (x,t,η) with $c_\tau > 0$.

The assumptions $H(j_\nu)$ and $H(j_\tau)$ are fairly general, see examples.

Problem formulation

We now introduce the spaces

$$V = \left\{v \in H^1(\Omega; \mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_D\right\},$$

$$H = L^2(\Omega; \mathbb{R}^d), \quad \mathcal{H} = L^2(\Omega; \mathcal{S}_d),$$

$\mathcal{S}_d$ is the space $\mathbb{R}_s^{d\times d}$ of symmetric matrices of order d,

$$\mathcal{V} = L^2(0, T; V), \quad \mathcal{W} = \{w \in \mathcal{V} \mid w' \in \mathcal{V}^*\}.$$

Let $A\colon (0, T) \times V \longrightarrow V^*$ and $B\colon V \longrightarrow V^*$ be defined by

$$\langle A(t, u), v\rangle_{V^*\times V} = \langle \mathcal{C}(x, t, \varepsilon(u)), \varepsilon(v)\rangle_{\mathcal{H}} \text{ for } u, v \in V,\ t \in (0, T),$$

$$\langle Bu, v\rangle_{V^*\times V} = \langle \mathcal{G}(x, \varepsilon(u)), \varepsilon(v)\rangle_{\mathcal{H}} \text{ for } u, v \in V.$$

We obtain the following hemivariational inequality formulation:

find $u\colon (0,T) \to V$ such that $u \in \mathcal{V}$, $u' \in \mathcal{W}$ and

$$\begin{cases} \langle u''(t) + A(t, u'(t)) + Bu(t), v\rangle_{V^* \times V} + \\ + \int_{\Gamma_C} \big(j_\nu^0(x,t,u_\nu(x,t); v_\nu(x)) + j_\tau^0(x,t,u_\tau'(x,t); v_\tau(x)) \big)\, d\Gamma \geq \\ \qquad \geq \langle f(t), v\rangle_{V^* \times V} \text{ for all } v \in V \text{ and a.e. } t \\ u(0) = u_0, \ \ u'(0) = u_1, \end{cases}$$

where

$$\langle f(t), v\rangle_{V^* \times V} = \langle f_1(t), v\rangle_H + \langle f_2(t), v\rangle_{L^2(\Gamma_N; \mathbb{R}^d)}$$

for all $v \in V$ and a.e. $t \in (0,T)$.

Abstract evolution inclusion and a hemivariational inequality

Let V and Z be separable Banach spaces and let H be a separable Hilbert space. Suppose

$$V \subset Z \subset H \subset Z^* \subset V^*$$

with dense and continuous embeddings and that $V \subset Z$ compactly. Define the spaces $\mathcal{V} = L^2(0,T;V)$, $\mathcal{Z} = L^2(0,T;Z)$ and $\mathcal{W} = \{v \in \mathcal{V} \mid v' \in \mathcal{V}^*\}$.

Consider the following second order evolution inclusion:

$$\begin{cases} u''(t) + A(t,u'(t)) + Bu(t) + F(t,u(t),u'(t)) \ni f(t) \text{ a.e. } t \\ u(0) = u_0,\ u'(0) = u_1. \end{cases}$$

Hypotheses

$\underline{H(A)}$: $A\colon (0,T)\times V \longrightarrow V^*$ is an operator such that

(i) $A(\cdot, v)$ is measurable on $(0,T)$ for all $v \in V$;

(ii) $A(t,\cdot)$ is pseudomonotone for every $t \in (0,T)$;

(iii) $||A(t,v)||_{V^*} \leq a(t) + b||v||$ with $a \in L^2(0,T)$, $a \geq 0$, $b > 0$;

(iv) $\langle A(t,v), v\rangle_{V^*\times V} \geq \alpha||v||^2$ for a.e. $t \in (0,T)$, for all $v \in V$ with $\alpha > 0$.

$\underline{H(B)}$: $B \in \mathcal{L}(V, V^*)$ is symmetric operator and coercive.

$\underline{H(F)}$: $F\colon (0,T) \times V \times V \to 2^{Z^*}$ has nonempty, convex and closed values and

(i) $F(\cdot, u, v)$ is measurable for all $u, v \in V$;

(ii) $F(t, \cdot, \cdot)$ is usc from $V \times V$ into w-Z^* for a.e. $t \in (0,T)$, where $V \times V$ is endowed with $(Z \times Z)$-topology;

(iii) $\|F(t,u,v)\|_{Z^*} \leq b_0(t) + b_1\|u\| + b_2\|v\|$ for $u, v \in V$ with $b_0 \in L^2(0,T)$, b_0, b_1, $b_2 \geq 0$.

$\underline{(H_0)}$: $f \in \mathcal{V}^*$, $u_0 \in V$, $u_1 \in H$.

$\underline{(H_1)}$: $\alpha > 2\sqrt{3}c_0(b_1T + b_2)$, where $c_0 > 0$ denotes the embedding constant of V into Z, i.e. $\|\cdot\|_Z \leq c_0\|\cdot\|$.

THEOREM 1 *Under the hypotheses $H(A)$, $H(B)$, $H(F)$, (H_0) and (H_1) the evolution inclusion admits a solution.*

We are now in a position to apply Theorem 1 to hemivariational inequality. We introduce functional $J\colon (0,T) \times L^2(\Gamma_C;\mathbb{R}^d)^2 \longrightarrow \mathbb{R}$ given by

$$J(t,u,z) = \int_{\Gamma_C} g(x,t,u(x),z(x))\, d\Gamma(x), \ \ t \in (0,T),$$

for $u, z \in L^2(\Gamma_C;\mathbb{R}^d)$,

where $g\colon \Gamma_C \times (0,T) \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is defined by

$$g(x,t,\xi,\eta) = j_\nu(x,t,\xi_\nu) + j_\tau(x,t,\eta_\tau)$$

for $(x,t) \in \Gamma_C \times (0,T)$ and $\xi, \eta \in \mathbb{R}^d$.

We introduce the following spaces:

$$\boldsymbol{V} \text{ is a closed subspace of } \boldsymbol{H}^1(\boldsymbol{\Omega};\mathbb{R}^d), \quad \boldsymbol{H} = \boldsymbol{L}^2(\boldsymbol{\Omega};\mathbb{R}^d),$$

$$\boldsymbol{Z} = \boldsymbol{H}^{\theta}(\boldsymbol{\Omega};\mathbb{R}^d) \text{ with a fixed } \boldsymbol{\theta} \in (\mathbf{1/2},\mathbf{1}).$$

It is well known that $\boldsymbol{V} \subset \boldsymbol{Z} \subset \boldsymbol{H} \subset \boldsymbol{Z}^* \subset \boldsymbol{V}^*$ densely and continuously, and $\boldsymbol{V} \subset \boldsymbol{Z}$ compactly.

Let

$$\gamma\colon \boldsymbol{Z} \longrightarrow \boldsymbol{L}^2(\Gamma_C;\mathbb{R}^d)$$

be the trace operator and let $\gamma^*\colon \boldsymbol{L}^2(\Gamma_C;\mathbb{R}^d) \longrightarrow \boldsymbol{Z}^*$ stands for its adjoint.

We define the multivalued mapping $F\colon (0,T)\times V\times V \longrightarrow 2^{Z^*}$ by

$$F(t,u,v) = S\,R^*\,\partial J(t,R(u,v)) \ \text{ for } u,v\in V,\ t\in(0,T),$$

where $S\colon Z^*\times Z^* \longrightarrow Z^*$ and $R\colon Z\times Z \longrightarrow L^2(\Gamma_C;\mathbb{R}^d)^2$ are given by $S(z_1,z_2) = z_1+z_2$, $R(u,v) = (\gamma u,\gamma v)$. The operator $R^*\colon L^2(\Gamma_C;\mathbb{R}^d)^2 \longrightarrow Z^*\times Z^*$ given by $R^*(z_1,z_2) = (\gamma^* z_1,\gamma^* z_2)$ is the adjoint of R and ∂J denotes the Clarke subdifferential of J with respect to (u,v).

PROPOSITION 2 *Let the multifunction F be as above and $H(\mathcal{C})$, $H(\mathcal{G})$, $H(f)$, $H(j_N)$, $H(j_T)$, (H_0) and (H_1) hold. Then every solution u to evolution inclusion*

$$\begin{cases} u''(t) + A(t,u'(t)) + Bu(t) + S\,R^*\,\partial J(t,R(u(t),u'(t))) \ni f(t) \\ u(0)=u_0,\ u'(0)=u_1 \end{cases}$$

is a solution to the hemivariational inequality.

Examples of the subdifferential boundary conditions of the type

$$-\sigma_\nu(t) \in \partial j_\nu(x,t,u_\nu(x,t)) \text{ on } \Gamma_C \times (0,T),$$
$$-\sigma_\tau(t) \in \partial j_\tau(x,t,u'_\tau(x,t)) \text{ on } \Gamma_C \times (0,T).$$

1^0) ***Contact with nonmonotone normal compliance.***
This is the case when $j_\nu\colon \mathbb{R} \longrightarrow \mathbb{R}$ is defined by (for simplicity we drop the (x,t)-dependence)

$$j_\nu(\xi) = \int_0^\xi p_\nu(\tau)\,d\tau, \quad \xi \in \mathbb{R},$$

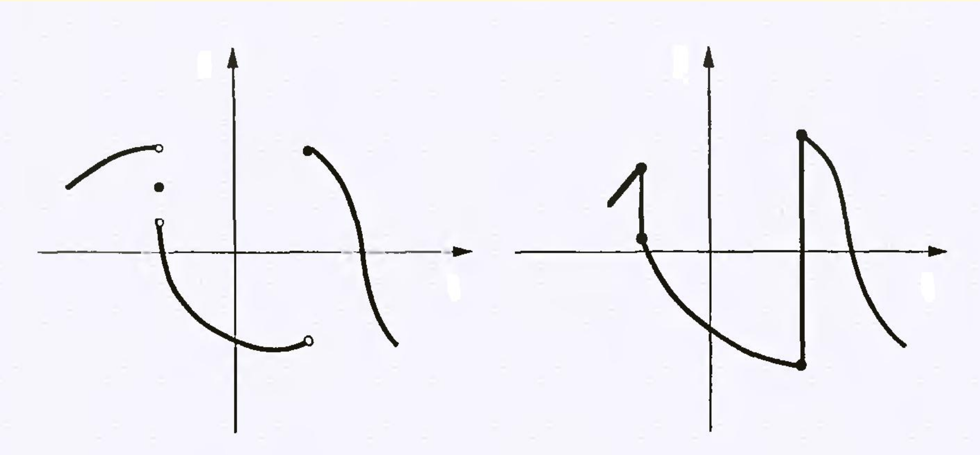
where $p_\nu\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $p_\nu \in L^\infty_{loc}(\mathbb{R})$, $|p_\nu(s)| \le p_1(1+|s|)$ for $s \in \mathbb{R}$ with $p_1 > 0$.

In this case $-\sigma_\nu \in \widehat{p_\nu}(u_\nu)$ on $\Gamma_C \times (0,T)$, where $\widehat{p_\nu}\colon \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ is obtained by filling in the jumps procedure.

Filling in the jumps procedure (1 dimensional)

Let $p_\nu\colon \mathbb{R} \longrightarrow \mathbb{R}$ be such that $p_\nu \in L^\infty_{loc}(\mathbb{R})$, $|p_\nu(s)| \le p_1(1 + |s|)$ for $s \in \mathbb{R}$ with $p_1 > 0$ and limits $\lim_{r\to s\pm} p_\nu(r)$ exist for all $s \in \mathbb{R}$. Define

$$j_\nu(r) = \int_0^r p_\nu(\tau)\, d\tau, \ \ r \in \mathbb{R}.$$



Then

$$-\sigma_\nu \in \partial j_\nu(u_\nu) = \widehat{p_\nu}(u_\nu) \text{ on } \Gamma_C \times (0, T),$$

where $\widehat{p_\nu}\colon \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ is obtained by filling in the jumps procedure.

Example 1.

Let $p_\nu \in L^\infty_{loc}(\mathbb{R})$ be given by

$$p_\nu(s) = \begin{cases} as & \text{if} \quad s \in (-\infty, -1) \cup (1, +\infty) \\ 2as & \text{if} \quad s \in (-1, 1), \end{cases}$$

where $a > 0$. Then $|p_\nu(s)| \leq a(1+|s|)$ for $s \in \mathbb{R}$ and the nonconvex function $j_\nu \colon \mathbb{R} \to \mathbb{R}$ can be expressed as a minimum of two convex functions, i.e. $j_\nu(s) = \min\{j_1(s), j_2(s)\}$, where $j_1(s) = as^2$ and $j_2(s) = \frac{a}{2}(s^2 + 1)$ for $s \in \mathbb{R}$. Then

$$\partial j_\nu(s) = \begin{cases} as & \text{if} \quad s \in (-\infty, -1) \cup (1, +\infty) \\ 2as & \text{if} \quad s \in (-1, 1) \\ [a, 2a] & \text{if} \quad s = 1 \\ [-2a, -a] & \text{if} \quad s = -1. \end{cases}$$

This model example can be modified to obtain nonmonotone zig-zag relations which describe the adhesive contact problems and contact laws for a granular material and a reinforced concrete, cf. Panagiotopoulos (1993), Naniewicz and Panagiotopoulos (1995), Goeleven, Motreanu, Dumont and Rochdi (2003).

Example 2.

We consider the nonmonotone Winkler law. Let $p_\nu \in L^\infty_{loc}(\mathbb{R})$ be defined by

$$p_\nu(s) = \begin{cases} 0 & \text{if } s \in (-\infty, 0) \cup (e, +\infty) \\ k_0 s & \text{if } s \in (0, e), \end{cases}$$

where e is a small positive constant and $k_0 > 0$ is the Winkler coefficient. Then $|p_\nu(s)| \leq k_0 e$ for $s \in \mathbb{R}$ and $j_\nu(s) = \min\{g_1(s), g_2(s)\}$, where

$$g_1(s) = \begin{cases} 0 & \text{if } \quad s < 0 \\ \dfrac{k_0}{2} s^2 & \text{if } \quad s \geq 0 \end{cases}$$

and $g_2(s) = \frac{k_0}{2} e^2$ for $s \in \mathbb{R}$. Assuming that the tangential forces are known $\sigma_\tau = C_\tau$, $C_\tau = C_\tau(x)$ is given on $\Gamma_C \times (0, T)$, the condition $-\sigma_\nu \in \partial j_\nu(u_\nu)$ reduces to the Winkler law.

If additionally p_ν is a continuous function, then the normal compliance condition reduces to the equation

$$-\sigma_\nu = p_\nu(u_\nu) \text{ on } \Gamma_C \times (0,T)$$

studied by Anderson (1995), Kikuchi and Oden (1988), Klarbring, Mikelic and Shillor (1988), Rochdi, Shillor and Sofonea (1998), Han and Sofonea (2002).

The friction law $-\sigma_\tau(t) \in \partial j_\tau(x,t,u'_\tau(x,t))$ is a generalization of

- the frictionless contact if $j_\tau = 0$,
- a version of Tresca's friction law if $j_\tau(x,t,\xi) = h(x,t)\|\xi\|$, where $h \in L^\infty(\Gamma_C \times (0,T))$ and $h > 0$, Amassad and Fabre (2002), Naniewicz and Panagiotopoulos (1995).

2^0) *Contact with simplified Coulomb's friction law.*

Consider a contact problem modeled by a simplified version of Coulomb's law (Duvaut and Lions (1976), Panagiotopoulos (1985,1993), Han and Sofonea (2002)):

$$\begin{cases} -\sigma_\nu = S, \ \ \|\sigma_\tau\| \le \mu|\sigma_\nu| \text{ with} \\ \ \ \|\sigma_\tau\| < \mu|\sigma_\nu| \implies u'_\tau = 0, \\ \ \ \|\sigma_\tau\| = \mu|\sigma_\nu| \implies \exists\, \lambda \ge 0 : \sigma_\tau = -\lambda u'_\tau \text{ on } \Gamma_C \times (0,T). \end{cases}$$

Here $S \in L^\infty(\Gamma_C)$ is a given normal stress on Γ_C and $\mu \in L^\infty(\Gamma_C)$, $\mu \ge 0$ a.e. on Γ_C is the coefficient of friction.

In this case $j_\nu(x,t,\xi) = -S(x)\xi$ for $(x,t,\xi) \in \Gamma_C \times (0,T) \times \mathbb{R}$ and $j_\tau(x,t,\eta) = -\mu(x)S(x)\|\eta\|_{\mathbb{R}^d}$ for $(x,t,\eta) \in \Gamma_C \times (0,T) \times \mathbb{R}^d$. We have $\partial j_\nu(x,t,\xi) = -S(x)$ and

$$\partial j_\tau(x,t,\eta) = \begin{cases} -\mu(x)S(x)\overline{B}(0,1) & \text{if } \eta = 0 \\ -\mu(x)S(x)\dfrac{\eta}{\|\eta\|} & \text{if } \eta \ne 0, \end{cases}$$

where $\overline{B}(0,1)$ is the unit closed ball in $\mathbb{R}^d$.

Contact problems with other boundary conditions

Consider the problem:

$$
\begin{cases}
u''(t) - \operatorname{div}\sigma(u(t),u'(t)) = f_1(t) & \text{in } Q\\
\sigma(u,u') = \mathcal{C}(\varepsilon(u')) + \mathcal{G}(\varepsilon(u)) & \text{in } Q\\
u = 0 & \text{on } \Gamma_D \times (0,T)\\
\sigma n = f_2 & \text{on } \Gamma_N \times (0,T)\\
-\sigma_\nu(t) \in \partial j_\nu(x,t,u'_\nu(x,t)) & \text{on } \Gamma_C \times (0,T)\\
-\sigma_\tau(t) \in \partial j_\tau(x,t,u'_\tau(x,t)) & \text{on } \Gamma_C \times (0,T)\\
u(0) = u_0, \;\; u'(0) = u_1 & \text{in } \Omega.
\end{cases}
$$

We obtain the following hemivariational inequality:
find $u\colon (0,T) \longrightarrow V$ such that $u \in \mathcal{V}$, $u' \in \mathcal{W}$ and

$$
\begin{cases}
\langle u''(t) + A(t,u'(t)) + Bu(t), v\rangle_{V^*\times V} + \\
+ \displaystyle\int_{\Gamma_C} \left(j^0_\nu(x,t,u'_\nu(x,t); v_\nu(x)) + j^0_\tau(x,t,u'_\tau(x,t); v_\tau(x))\right) d\Gamma \geq \\
\qquad \geq \langle f(t), v\rangle_{V^*\times V} \;\text{ for all } v \in V \text{ and a.e. } t\\
u(0) = u_0, \;\; u'(0) = u_1.
\end{cases}
$$

Now, we introduce functional $\boldsymbol{H} : (0, T) \times L^2(\Gamma_C; \mathbb{R}^d) \longrightarrow \mathbb{R}$ defined by

$$\boldsymbol{H}(t, v) = \int_{\Gamma_C} (j_\nu(x, t, v_\nu(x)) + j_\tau(x, t, v_\tau(x)))\, d\Gamma$$

for all $t \in (0, T)$, $v \in L^2(\Gamma_C; \mathbb{R}^d)$.

We obtain the existence of solutions to the hemivariational inequality by using the evolution inclusion of the form

$$\begin{cases} \text{find } u \in \mathcal{V} \text{ with } u' \in \mathcal{W} \text{ such that} \\ u''(t) + A(t, u'(t)) + Bu(t) + \gamma^* \partial H(t, \gamma u'(t)) \ni f(t) \quad \text{a.e. } t \\ u(0) = u_0,\ u'(0) = u_1, \end{cases}$$

where ∂H stands for a subdifferential of $H(t, \cdot)$.

Uniqueness of solutions

We need additional hypotheses.

$\underline{H(A)_1}$: $\quad A\colon (0,T)\times V \to V^*$ satisfies $H(A)$ and

$$\langle A(t,u) - A(t,v), u - v\rangle_{V^*\times V} \ge m_1 \|u-v\|^2$$

for all u, $v \in V$ and a.e. $t \in (0,T)$ with $m_1 > 0$

$\underline{H(H)_1}$: $\quad H\colon (0,T)\times L^2(\Gamma_C;\mathbb{R}^d) \to \mathbb{R}$ satisfies the previous assumptions and the following relaxed monotonicity condition

$$\langle z_1 - z_2, w_1 - w_2\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} \ge -m_2 \|w_1 - w_2\|^2_{L^2(\Gamma_C;\mathbb{R}^d)}$$

for all $z_i \in \partial H(t, w_i)$, $w_i \in L^2(\Gamma_C;\mathbb{R}^d)$, $i = 1, 2$
and a.e. $t \in (0,T)$ with $m_2 > 0$.

$\underline{(H_2)}$: $\quad m_1 > m_2 \, \|\gamma\|^2$.

PROPOSITION 3 *Under the additional hypotheses, the hemivariational inequality admits a unique solution.*

Examples of the subdifferential boundary conditions of the type

$$-\sigma_\nu(t) \in \partial j_\nu(x,t,u'_\nu(x,t)) \text{ on } \Gamma_C \times (0,T),$$
$$-\sigma_\tau(t) \in \partial j_\tau(x,t,u'_\tau(x,t)) \text{ on } \Gamma_C \times (0,T).$$

3^0) *Contact with nonmonotone normal damped response.*

In this case $j_\nu(\xi) = \int_0^\xi p_\nu(\tau)\,d\tau$, $\xi \in \mathbb{R}$, where $p_\nu \in L^\infty_{loc}(\mathbb{R})$ is a prescribed function and $-\sigma_\nu \in \widehat{p_\nu}(u'_\nu)$ on $\Gamma_C \times (0,T)$. If in addition p_ν is a continuous function, then the condition reduces to $-\sigma_\nu = p_\nu(u'_\nu)$ on $\Gamma_C \times (0,T)$, Awbi et al. (2000).

If $p_\nu(s) = k_1 s$ with $k_1 > 0$, we obtain $-\sigma_\nu = k_1 u'_\nu$ on $\Gamma_C \times (0,T)$ which means that the resistance of the foundation to penetration is proportional to the normal velocity and models the motion of a deformable body on a support of granular material, Sofonea and Shillor (2001).

If $p_\nu(s) = k_2 s_+ + k_3$, where $s_+ = \max\{0,s\}$ with $k_2 > 0$ and $k_3 \geq 0$, we get the model in which the contact surface is supposed to be covered with a lubricant that contains solid particles, Rochdi, Shillor and Sofonea (1998).

If p_ν and p_τ are continuous and monotone, we have a model studied in the dynamic case in Chau, Han and Sofonea (2002).

4^0) *Viscous contact with Tresca's friction law.*

We consider a model of damped response contact with time dependent Tresca's friction law. We use the following boundary conditions:

$$\begin{cases} -\sigma_\nu = k(x)|u'_\nu|^{q-1}u'_\nu \\ \|\sigma_\tau\| \le h(t) \text{ with} \\ \quad \|\sigma_\tau\| < h(t) \implies u'_\tau = 0, \\ \quad \|\sigma_\tau\| = h(t) \implies \exists\, \lambda \ge 0 : \sigma_\tau = -\lambda u'_\tau \quad \text{on } \Gamma_C \times (0,T), \end{cases}$$

where $k \in L^\infty(\Gamma_C)$, $k > 0$ a.e. on Γ_C, $0 < q \le 1$, $h \in L^\infty(\Gamma_C \times (0,T))$ and $h > 0$ a.e. on $\Gamma_C \times (0,T)$.

In this case, we have $j_\tau(x,t,\xi) = h(x,t)\|\xi\|$ and $j_\nu(x,t,s) = \dfrac{k(x)}{q+1}|s|^{q+1}$. Thus $\partial j_\nu(x,t,s) = k(x)|s|^{q-1}s$,

$$\partial j_\tau(x,t,\xi) = \begin{cases} h(x,t)\overline{B}(0,1) & \text{if } \xi = 0 \\ h(x,t)\dfrac{\xi}{\|\xi\|} & \text{if } \xi \ne 0, \end{cases}$$

cf. Duvaut and Lions (1976), Panagiotopoulos (1985), Amassad and Sofonea (1998), Shillor and Sofonea (2000), Han and Sofonea (2002), Amassad and Fabre (2002).

5^0) *Viscous contact with power-law friction boundary conditions.*

In this case the function j_ν is as in 4^0) and j_τ is given by

$$j_\tau(x, t, \xi) = \frac{\mu(x)}{p+1}\|\xi\|^{p+1},$$

where $\mu \in L^\infty(\Gamma_C)$, $\mu > 0$ a.e. on Γ_C and $0 < p \leq 1$.

This choice leads to the following boundary conditions

$$\begin{cases} -\sigma_\nu = k(x)|u'_\nu|^{q-1}u'_\nu \\ -\sigma_\tau = \mu(x)\|u'_\tau\|^{p-1}u'_\tau \end{cases} \text{ on } \Gamma_C \times (0, T).$$

cf. Han and Sofonea (2002).

6^0) The choice $j_\nu(x,t,s) = \frac{1}{2}k(x)(s_+)^2 + p_0 s$ and $j_\tau(x,t,\xi) = h(x,t)\|\xi\|$ with positive functions k, $p_0 \in L^\infty(\Gamma_C)$ and $h \in L^\infty(\Gamma_C \times (0,T))$, leads to *the normal damped response with time dependent Tresca's friction law*

$$\begin{cases} -\sigma_\nu = k(x)(u'_\nu)_+ + p_0 \\ \\ \|\sigma_\tau\| \leq h(t) \text{ with} \\ \quad \|\sigma_\tau\| < h(t) \implies u'_\tau = 0, \\ \quad \|\sigma_\tau\| = h(t) \implies \exists\, \lambda \geq 0 : \sigma_\tau = -\lambda u'_\tau \quad \text{on } \Gamma_C \times (0,T). \end{cases}$$

7^0) The choice $j_\nu(x,t,s) = \dfrac{1}{2}k(x)(s_+)^2 + p_0 s$ and $j_\tau(x,t,\xi) = \dfrac{\mu(x)}{p+1}\|\xi\|^{p+1}$ where k, p_0, $\mu \in L^\infty(\Gamma_C)$ are positive functions, $0 < p \le 1$, leads to *the normal damped response with power-law friction*

$$\begin{cases} -\sigma_\nu = k(x)(u'_\nu)_+ + p_0 \\ -\sigma_\tau = \mu(x)\|u'_\tau\|^{p-1}u'_\tau \quad \text{on } \Gamma_C \times (0,T), \end{cases}$$

cf. Han and Sofonea (2002).

Remarks on other contact problems

The contact problem with the boundary conditions on $\Gamma_C \times (0,T)$:

$$-\sigma_\nu(t) \in \partial j_\nu(x,t,u_\nu(x,t))$$

$$-\sigma_\tau(t) \in \partial j_\tau(x,t,u_\tau(x,t))$$

leads to the following hemivariational inequality:

find $u\colon (0,T) \to V$ such that $u \in \mathcal{V}$, $u' \in \mathcal{W}$ and

$$\begin{cases} \langle u''(t) + A(t,u'(t)) + Bu(t), v\rangle_{V^*\times V} + \\ + \displaystyle\int_{\Gamma_C} \big(j_\nu^0(x,t,u_\nu(x,t);v_\nu(x)) + j_\tau^0(x,t,u_\tau(x,t);v_\tau(x))\big)\, d\Gamma \geq \\ \qquad \geq \langle f(t), v\rangle_{V^*\times V} \text{ for all } v \in V \text{ and a.e. } t \\ u(0) = u_0, \ \ u'(0) = u_1. \end{cases}$$

The existence result for the above hemivariational inequality is established.

The results on contact problem with the following boundary conditions on $\Gamma_C \times (0, T)$:

$$-\sigma_\nu(t) \in \partial j_\nu(x, t, u_\nu(x, t))$$

$$-\sigma_\tau(t) \in \partial j_\tau(x, t, u_\tau(x, t))$$

generalize the existence for static, quasistatic and dynamic models obtained by

- Mistakidis and Panagiotopoulos (1998) (the reaction–displacement diagrams which are the nonmonotone variants of the friction law of Coulomb)
- Dumont, Goeleven, Rochdi, Kuttler and Shillor (2000) (models in geomechanics and rock interface analysis, and models with friction laws between reinforcement and concrete in concrete structures)
- Panagiotopoulos (1993) (the sawtooth laws generated by nonconvex superpotentials describing the partial cracking and crushing of the adhesive bonding material).

Application to a bilateral contact problem

We have obtained the result on the existence and uniqueness of weak solutions to the hemivariational inequality for the problem

$$\begin{cases} u''(t) - \operatorname{div} \sigma(u(t), u'(t)) = f_1(t) & \text{in } Q \\ \sigma(u, u') = \mathcal{C}(\varepsilon(u')) + \mathcal{G}(\varepsilon(u)) & \text{in } Q \\ u = 0 & \text{on } \Gamma_D \times (0, T) \\ \sigma\nu = f_2 & \text{on } \Gamma_N \times (0, T) \\ u_\nu = 0 & \text{on } \Gamma_C \times (0, T) \\ \|\sigma_\tau\| \le h(t) \text{ with} & \\ \quad \begin{array}{l} \|\sigma_\tau\| < h(t) \implies u'_\tau = 0, \\ \|\sigma_\tau\| = h(t) \implies \exists\, \lambda \ge 0 : \ \sigma_\tau = -\lambda u'_\tau \end{array} & \text{on } \Gamma_C \times (0, T) \\ u(0) = u_0, \ \ u'(0) = u_1 & \text{in } \Omega. \end{cases}$$

For the bilateral contact problem the hemivariational inequality reads as follows:

find $u\colon (0,T) \longrightarrow V_0$ such that $u \in \mathcal{V}_0$, $u' \in \mathcal{W}_0$ and

$$\begin{cases} \langle u''(t) + A(t, u'(t)) + Bu(t), v\rangle_{V_0^* \times V_0} + \\ + \int_{\Gamma_C} j_\tau^0(x, t, u'_\tau(x,t); v_\tau(x))\, d\Gamma \geq \\ \qquad\qquad \geq \langle f(t), v\rangle_{V_0^* \times V_0} \text{ for all } v \in V_0 \text{ and a.e. } t \\ u(0) = u_0, \ \ u'(0) = u_1, \end{cases}$$

where $V_0 = \{v \in H^1(\Omega;\mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_D,\ v_\nu = 0 \text{ on } \Gamma_C\}$, $\mathcal{V}_0 = L^2(0,T;V_0)$, $\mathcal{W}_0 = \{w \in \mathcal{V}_0 \mid w' \in \mathcal{V}_0^*\}$ and the function $j_\tau\colon \Gamma_C \times (0,T) \times \mathbb{R}^d \longrightarrow \mathbb{R}$ given by $j_\tau(x,t,\xi) = h(x,t)\|\xi\|$.

We generalize Duvaut and Lions (1976), Panagiotopoulos (1985), Amassad and Sofonea (1998), Kuttler and Shillor (2001), Han and Sofonea (2002) and give an alternative approach to Kuttler and Shillor (1999).

A unified approach to dynamic contact problems in viscoelasticity

The approach is general and allows to unify several methods for models considered in contact mechanics and obtain new existence results which are not available in the literature.

The following multivalued boundary conditions for stationary, quasistatic and evolution problems can be studied by this approach:

- ⋆ the nonmonotone normal compliance condition
- ⋆ the simplified Coulomb friction law
- ⋆ the nonmonotone normal damped response condition
- ⋆ the viscous contact with Tresca's friction law
- ⋆ the viscous contact with power-law friction boundary conditions
- ⋆ the normal damped response with time dependent Tresca's friction
- ⋆ the normal damped response with power-law friction
- ⋆ the nonmonotone variants of the friction law of Coulomb
- ⋆ the sawtooth laws generated by nonconvex superpotentials.

Moreover, an existence result for the dynamic bilateral contact problem with time dependent Tresca's friction condition is obtained.

Recent advances in problems with hemivariational inequalities

- Existence and uniqueness of solutions
  - first order evolution problems
  - second order evolution problems
- Variational models of mechanics
  - elasticity
  - viscoelasticity
  - thermoviscoelasticity
  - models with hysteresis
  - heat conduction
  - fluid flow problems
  - piezoelectricity
- Homogenization of hemivariational inequalities
- Optimal control problems, shape optimization problems
- Inverse and identification problems

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Thank you !

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