european social fund in the czech republic
EUROPEAN UNION
MINISTRY OF EDUCATION, YOUTH AND SPORTS
OP Education for Competitiveness
INVESTMENTS IN EDUCATION DEVELOPMENT

Streamlining the Mathematics Studies at the Faculty of Science of Palacky University in Olomouc

Registry number: CZ.1.07/2.2.00/28.0141

Nonlinear Inclusions and Hemivariational Inequalities with Applications to Contact Mechanics. Part I: Static and Quasistatic Problems

Stanislaw Migórski

Jagiellonian University

Faculty of Mathematics and Computer Science

Kraków, Poland

migorski@uj.edu.pl

Outline of the talk

- Physical background, motivation and applications
- Elliptic hemivariational inequality
- Example: the destruction support problem
- Variational formulation and existence results
- Three models and three hemivariational inequalities
  - An elastic frictional problem
  - A viscoelastic frictional problem
  - An electro-elastic frictional problem

Based on some recent results from the following recent monograph.

S. Migorski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Advances in Mechanics and Mathematics, vol. **26**, Springer, New York, 2013.

Motivation: inequality problems in Mechanics

Convex energy functions $\Longrightarrow$ Monotone relations
$\Longrightarrow$ Variational Inequalities
(Signorini, Fichera, Duvaut, Lions)

Nonconvex energy functions $\Longrightarrow$ Nonmonotone relations
$\Longrightarrow$ Hemivariational Inequalities
(Panagiotopoulos, Naniewicz)

Motivation

- **Contact phenomena appear in everyday life and play a very important role in engineering structures and systems.**
- **These include: friction, wear, adhesion, frictional heat generation, and lubrication, among others; are inherently complex and time dependent; take place on the outer surfaces of parts and components, and involve thermal, physical and chemical processes.**
- **The need for a comprehensive well posed mathematical theory, based on fundamental physical principles, that can predict reliably the evolution of the contact process in different situations and under various conditions, has been recognized in recent years.**

Applications

In applications there are four main areas of friction control:

- **Low friction lubrication:** **needed in machinery to reduce wear, tear, and loss of useful energy.**
- **High friction:** **needed in brakes of cars, trains, and moving systems.**
- **Friction within specified bounds:** **needed in braking systems to avoid jerks and sudden accelerations and decelerations.**
- **Slip dependence:** **needed to avoid slip/stick transitions and the associated unpleasant noise - squeaks and squeals.**

Brakes

- **Brakes are used to transmit forces to reduce the velocity of a vehicle.**
- **If the friction coefficient reaches a certain value, the brake will become unstable and a squealing noise can occur.**
- **Industry is interested in reducing the squealing noise and the temperature development within the brakes. Otherwise cooling devices would have to be installed.**

Machine Tools

- In machine tool with friction (e.g. a grinding machine) the friction plays a role to develop smooth surfaces.
- The worn particles have high temperatures.
- One uses cooling fluid to reduce temperature, remove worn material from the grinding disk and increase the surface quality.
- Machine chattering can occur. Chatter vibrations belong to friction induced vibrations.

Motors

- Friction and wear problems between piston and cylinder of a motor are a dynamical contact problem.
- The oil acts as a lubricant within the contact regions and reduces friction and wear.
- Industry is interested in increasing the lifetime of motor components and optimize the system behavior.

Turbines

- **Turbine blades are excited by fluctuating gas forces.**
- **Friction is introduced to dissipate the vibration energy.**
- **Goal: to increase the lifetime of the blades and to reduce the vibration amplitudes and noise.**
- **Efficient contact models have to be developed for optimizing these structures.**

Bearings

- To increase the efficiency of slide and ball bearings, the bearing friction has to be lowered and the lubrication is used.
- The oil film on the ball bearings reduces friction forces and, hence, the wear.
- This is a multibody and multicontact problem with friction.
- Goal: to determine the longtime behavior of ball bearings in connection with the surroundings.

Wheel-rail contact

- **The whell-rail rolling contact problem is a typical example for friction used to transmit forces.**
- **The contact behavior depends on the material properties of the contacting bodies.**
- **The development of heat within the rolling contact influences the tangential contact forces.**
- **The development of wear leads to unround wheels, which increses the generation of noise and the cost of maintenance.**

Summary: contact problems with friction

- **Two bodies have different surface profiles and different materials.**
- **Both bodies can vibrate and move spatially, which is described by displacements and velocities of both bodies in the so-called state space.**
- **Friction is always correlated with development of wear and heat.**
- **The heat generated and the temperature distribution affect the material parameters.**
- **The worn material can act as a lubricant (so-called third body).**

The Clarke subdifferential

Given a locally Lipschitz function $h\colon E \to \mathbb{R}$, where E is a Banach space, we define (Clarke (1983)):

- **the generalized directional derivative** of h at $x \in E$ in the direction $v \in E$ by

$$h^0(x;v) = \limsup_{y\to x,\ t\downarrow 0} \frac{h(y+tv)-h(y)}{t}.$$

- **the generalized gradient** of h at x by $\partial h(x)$, is a subset of a dual space E^* given by

$$\partial h(x) = \{\zeta \in E^* \mid h^0(x;v) \geq \langle \zeta, v\rangle_{E^*\times E} \text{ for all } v \in E\}.$$

The locally Lipschitz function h is called **regular** (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $h'(x;v)$ exists and satisfies $h^0(x;v) = h'(x;v)$ for all $v \in E$.

Example of nonmonotone contact problem in elasticity

Let $\boldsymbol{\Omega} \subset \mathbb{R}^3$ be occupied by a linear elastic body in its undeformed state. The boundary $\boldsymbol{\Gamma} = \partial\boldsymbol{\Omega}$ of $\boldsymbol{\Omega}$ consists of three open disjoint parts $\boldsymbol{\Gamma}_D$, $\boldsymbol{\Gamma}_N$ and $\boldsymbol{\Gamma}_C$, i.e. $\boldsymbol{\Gamma} = \overline{\boldsymbol{\Gamma}}_D \cup \overline{\boldsymbol{\Gamma}}_N \cup \overline{\boldsymbol{\Gamma}}_C$. A point $\boldsymbol{x} \in \boldsymbol{\Omega}$ is refereed to a fixed Cartesian coordinate system. We use the standard notation, for $\boldsymbol{i}$, $\boldsymbol{j} = \mathbf{1}, \mathbf{2}, \mathbf{3}$,

$\boldsymbol{u} = (\boldsymbol{u}_i)$	the displacement vector
$\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ij})$	the stress tensor
$\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}_{ij})$	the strain tensor
$\boldsymbol{f} = (\boldsymbol{f}_i)$	the volume force vector
$\boldsymbol{\nu} = (\boldsymbol{\nu}_i)$	the outward unit normal vector to $\boldsymbol{\Gamma}$

We also decompose the stress vector $\boldsymbol{\sigma}$ on $\boldsymbol{\Gamma}$ and the displacement $\boldsymbol{u}$ on $\boldsymbol{\Gamma}$ into the normal and tangential components:

$$\boldsymbol{\sigma}_\nu = (\boldsymbol{\sigma}\,\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\,\boldsymbol{\nu} - \boldsymbol{\sigma}_\nu\,\boldsymbol{\nu},$$

$$\boldsymbol{u}_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}, \qquad \boldsymbol{u}_\tau = \boldsymbol{u} - \boldsymbol{u}_\nu\,\boldsymbol{\nu}.$$

In the framework of the small deformation theory, we have the following pointwise relations in Ω:

$$\sigma_{ij,j}(u) = -f_i \quad \textbf{(the equilibrium equation)}$$

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \textbf{(the strain–displacement law)}$$

$$\sigma_{ij}(u) = c_{ijkl}\,\varepsilon_{kl}(u) \quad \textbf{(the constitutive equation, Hooke's law)}$$

where $\sigma_{ij,j}(u) = \frac{\partial}{\partial x_j}\sigma_{ij}(u)$ and the elasticity tensor $\{c_{ijkl}\}$ is assumed to satisfy the ellipticity and symmetry properties:

$$\begin{cases} c_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl} \geq c_0\,\varepsilon_{ij}\,\varepsilon_{ij} \ \text{ a.e. in } \Omega, \text{ for all } \varepsilon_{ij} = \varepsilon_{ji} \in \mathbb{R} \text{ with } c_0 > 0 \\ c_{ijkl} = c_{jikl} = c_{klij}. \end{cases}$$

In order to give a complete formulation of the problem we consider the following boundary conditions:

on Γ_D: $\boldsymbol{u} = \boldsymbol{U}$ (prescribed displacement),

on Γ_N: $\boldsymbol{\sigma\nu} = \boldsymbol{F}$ (i.e. $\boldsymbol{\sigma}_i = \boldsymbol{F}_i(\boldsymbol{x})$ are prescribed tractions),

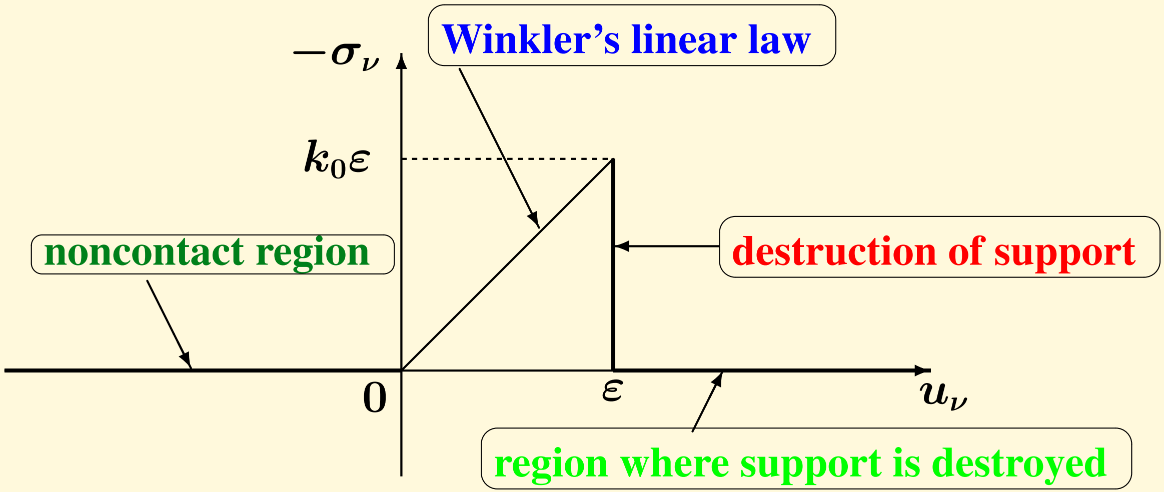
on Γ_C: $\boldsymbol{\sigma}_\tau = \boldsymbol{C}_\tau$ (the tangential forces $\boldsymbol{C}_\tau = \boldsymbol{C}_\tau(\boldsymbol{x})$ are known)

$\boldsymbol{\sigma}_\nu$ satisfies the idealized law

$$\begin{cases} \boldsymbol{\sigma}_\nu = 0 & \text{if} \quad \boldsymbol{u}_\nu < 0 \\ \boldsymbol{\sigma}_\nu + \boldsymbol{k}_0 \boldsymbol{u}_\nu = 0 & \text{if} \quad 0 \le \boldsymbol{u}_\nu < \varepsilon \\ -\boldsymbol{k}_0 \varepsilon \le \boldsymbol{\sigma}_\nu \le 0 & \text{if} \quad \boldsymbol{u}_\nu = \varepsilon \\ \boldsymbol{\sigma}_\nu = 0 & \text{if} \quad \boldsymbol{u}_\nu > \varepsilon. \end{cases}$$

For simplicity in the Dirichlet condition on Γ_D, we take $\boldsymbol{U} = 0$.

The constant $\boldsymbol{k}_0 > 0$ is called the Winkler coefficient.



Nonmonotone diagram for the Winkler-type support

The Winkler coefficient $k_0 > 0$ and $\varepsilon > 0$.

In the **noncontact region** $\sigma_\nu = 0$, in the **contact region** the support generates a reaction force which is proportional to its deformation $-\sigma_\nu \sim u_\nu$, the **destruction of the support appears when the tractions reach the limited value** (the maximal value of reactions that can be maintained by the support in $k_0\varepsilon$) and again $\sigma_\nu = 0$ **in the region where the support is destroyed**.

Since the diagram $(u_\nu, -\sigma_\nu)$ is **nonmonotone**, it is not possible to formulate the destruction support problem as a variational inequality.

In order to give the variational formulation of the above problem, we introduce the function $\beta\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\beta(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ k_0 t & \text{if} \quad t \in [0, \varepsilon) \\ 0 & \text{if} \quad t \geq \varepsilon \end{cases}$$

and define the multivalued map $\widehat{\beta}\colon \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ which is obtained from β by filling in the jump at $t = \varepsilon$.

Given $\beta \in L^\infty_{loc}(\mathbb{R})$ we define $\widehat{\beta}$ as follows

$$\widehat{\beta}(\xi) = [\underline{\beta}(\xi), \overline{\beta}(\xi)] \subset \mathbb{R},$$

where

$$\underline{\beta}(\xi) = \lim_{\delta \to 0^+} \operatorname*{ess\,inf}_{|t-\xi| \leq \delta} \beta(t), \quad \overline{\beta}(\xi) = \lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|t-\xi| \leq \delta} \beta(t).$$

It is known, Chang (1981), that there exists a locally Lipschitz function $j\colon \mathbb{R} \longrightarrow \mathbb{R}$ determined (up to an additive constant) by the relation $j(t) = \int_0^t \beta(s)\,ds$ and

$$\partial j(t) \subset \widehat{\beta}(t).$$

Additionaly, if $\lim_{t\to\xi^\pm} \beta(t)$ exist for every $\xi \in \mathbb{R}$, then we have

$$\partial j(t) = \widehat{\beta}(t) \text{ for } t \in \mathbb{R}.$$

Taking the above into consideration, the boundary condition given by Winkler's law can be written as

$$-\sigma_\nu \in \widehat{\beta}(u_\nu) = \partial j(u_\nu) \text{ on } \Gamma_C,$$

where $j\colon \mathbb{R} \longrightarrow \mathbb{R}$ is of the form

$$j(t) = \int_0^t \beta(s)\,ds = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2} k_0 t^2 & \text{if } 0 \le t \le \varepsilon \\ \frac{1}{2} k_0 \varepsilon^2 & \text{if } t > \varepsilon. \end{cases}$$

Let us introduce the space $\boldsymbol{V}$ of kinematically admissible displacements

$$\boldsymbol{V} = \{\, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega; \mathbb{R}^3) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D \,\}$$

and $\boldsymbol{H} = \boldsymbol{L}^2(\Omega; \mathbb{R}^3)$. Then, we have an evolution triple of spaces $(\boldsymbol{V}, \boldsymbol{H}, \boldsymbol{V}^*)$.

Let $\boldsymbol{v} \in \boldsymbol{V}$. Multiplying the equilibrium equation by $\boldsymbol{v}$, integrating over Ω and applying the Green formula we have

$$\sum_{i,j=1}^{3} \int_\Omega \sigma_{ij}(u) \frac{\partial v_i}{\partial x_j}\, dx =$$

$$= \sum_{i=1}^{3} \int_\Omega f_i\, v_i\, dx + \sum_{i,j=1}^{3} \int_\Gamma \sigma_{ij}(u)\, v_i\, \nu_j\, d\Gamma.$$

Taking into account the boundary conditions, symmetry of the stress tensor, we arrive at the problem

$$a(u,v) + \int_{\Gamma_C} (-\sigma_\nu)\, v_\nu \, d\Gamma = \langle l, v\rangle \ \text{ for all } \ v \in V,$$

where

$$a(u,v) = \sum_{i,j=1}^{n} \int_\Omega \sigma_{ij}(u)\, \varepsilon_{ij}(v)\, dx,$$

$$\langle l, v\rangle = \sum_{i=1}^{n} \int_\Omega f_i\, v_i\, dx + \sum_{i=1}^{n} \int_{\Gamma_N} F_i\, v_i\, d\Gamma + \sum_{i=1}^{n} \int_{\Gamma_C} C_{\tau i}\, v_{\tau i}\, d\Gamma.$$

By the definition of the Clarke subdifferential, we have

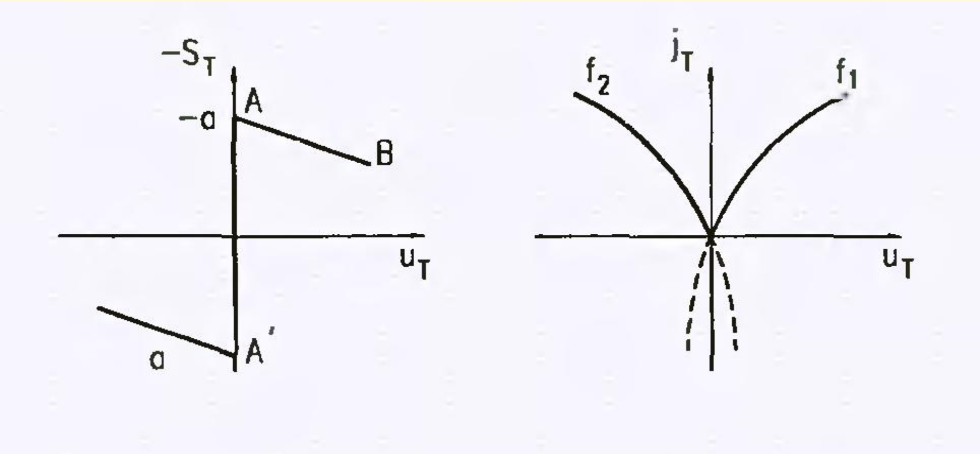
$$j^0(u_\nu; v_\nu) \geq -\sigma_\nu\, v_\nu \ \text{ for all } \ v \in V.$$

We obtain the problem: find $u \in V$ such that

$$a(u,v) + \int_{\Gamma_C} j^0(u_\nu; v_\nu)\, d\Gamma \geq \langle l, v\rangle \ \text{ for all } \ v \in V.$$

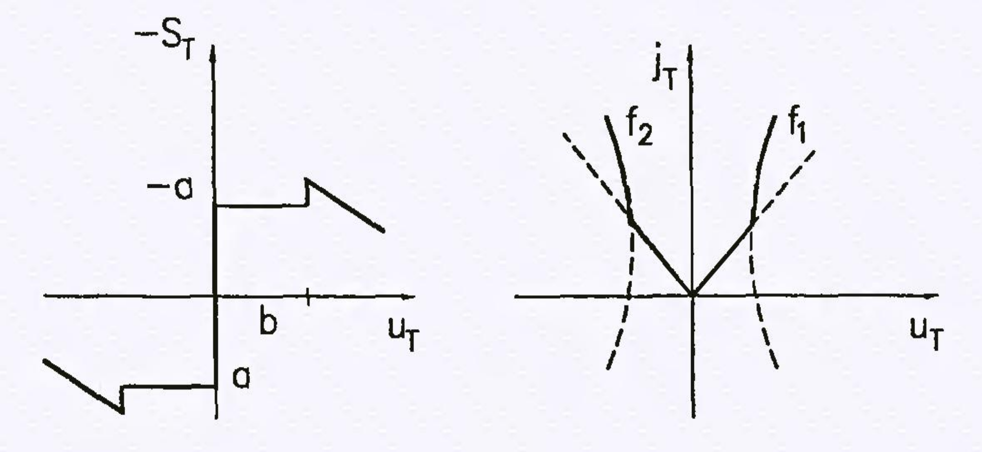
This problem is called **hemivariational inequality**.

Example: nonconvex friction law



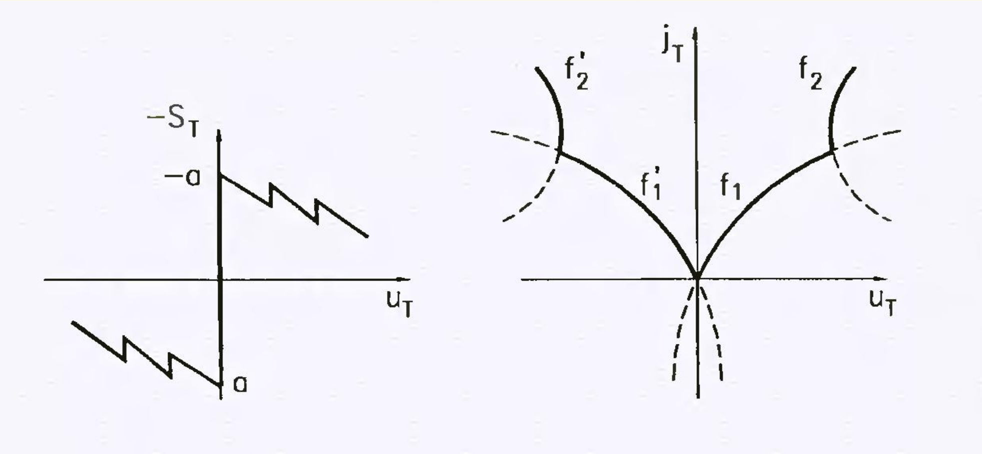
$$j_T(u_T) = \max\{f_1(u_T), f_2(u_T)\}$$

Example: nonconvex friction law



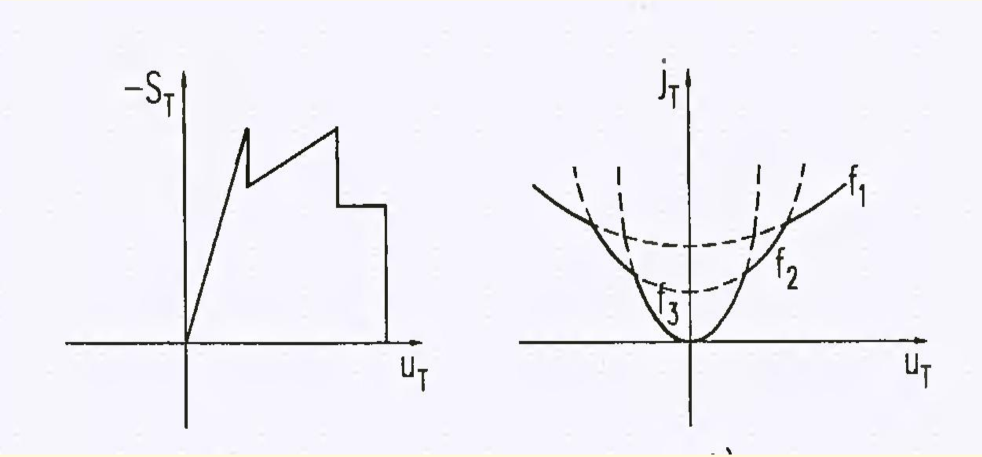
$$j_T(u_T) = \max\{a|u_T|, f_1(u_T), f_2(u_T)\}$$

Example: zig-zag friction law



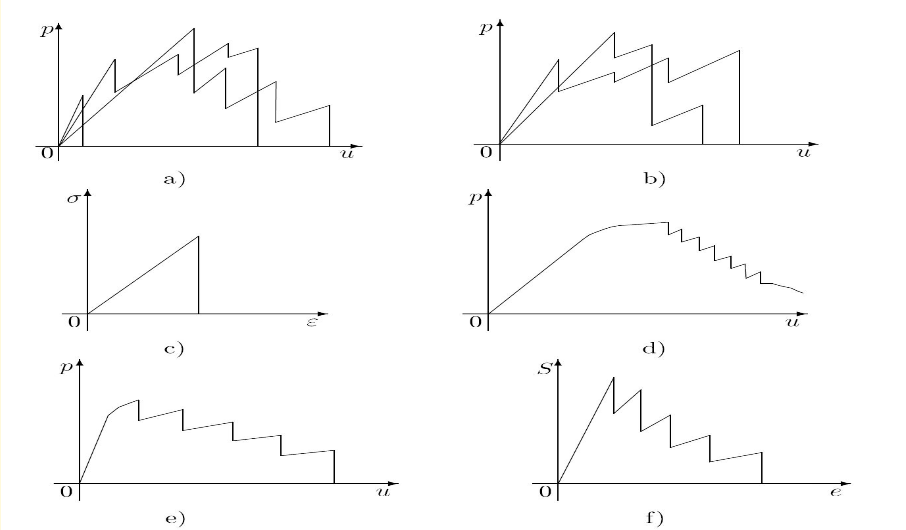
$$j_T(u_T) = \max\{f_1(u_T), f_2(u_T), f_1'(u_T), f_2'(u_T)\}$$

Example: zig-zag friction law



$$j_T(u_T) = \min\{f_1(u_T), f_2(u_T), f_3(u_T)\}$$

Nonmonotone laws



a) Force-displacement diagrams for laminated products
b) Force-displacement diagrams for glass fiber-reinforced epoxy laminated composites
c) Ply stress-strain curve in a lamina with brittle behavior
d) Force-displacement diagram for a graphite-epoxy composite laminate
e) Force-displacement diagram for an aluminium-beryllium composite beam
f) Scanlon's diagram

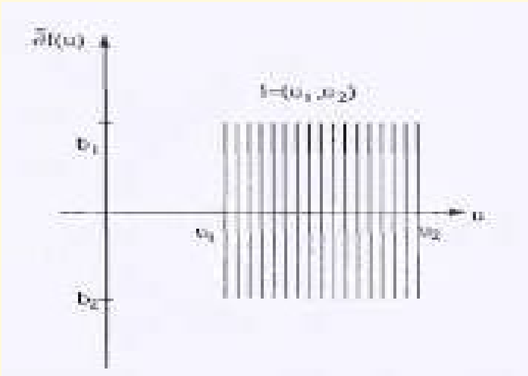
Example: infinite number of jumps

Let l be an open subset of the real line $\mathbb{R}$ and let M be a measurable subset of l such that for every open and nonempty subset I of l, $\operatorname{meas}(I \cap M) > 0$ and $\operatorname{meas}(I \cap (l \setminus M)) > 0$.
Let

$$g(s) = \begin{cases} b_1 & \text{if } s \in M \\ -b_2 & \text{if } s \notin M \end{cases}$$

and $j(r) = \int_0^r g(\theta)\, d\theta$. Then the nonconvex potential j is locally Lipschitz and

$$\partial j(r) = [-b_2, b_1] \text{ for every } r \in l.$$

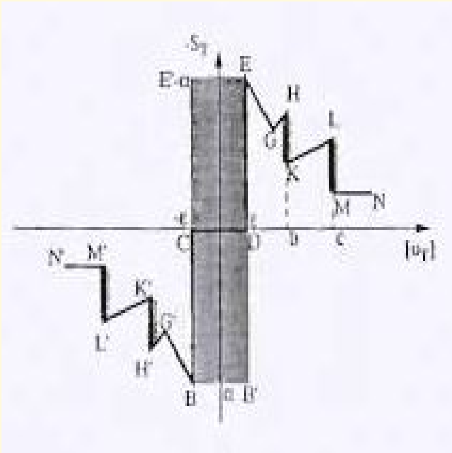


Example: fuzzy laws

Nonconvex potentials of the form

$$\partial j(r) = [-b_2, b_1] \text{ for every } r \in l = (-\varepsilon, \varepsilon)$$

allow to consider nonfully determined regions around the complete vertical segments of nonmonotone law. This type of laws are called ”fuzzy laws” after Panagiotopoulos (1993).



The expression of such nonfully determined laws in terms of nonconvex potentials permits the formulation of a variational theory for this class of problems.

Remark 1

A similar hemivariational inequality can be obtained when on Γ_C a pointwise boundary condition for the tangential force

$$-\boldsymbol{\sigma}_\tau \in \partial \boldsymbol{j}(\boldsymbol{u}_\tau) \quad \text{on} \quad \Gamma_C$$

is assumed and the normal forces $\boldsymbol{\sigma}_\nu = \boldsymbol{C}_\nu$, $\boldsymbol{C}_\nu = \boldsymbol{C}_\nu(\boldsymbol{x})$ are prescribed.

Such relation between $\boldsymbol{u}_\tau$ and $-\boldsymbol{\sigma}_\tau$ appears, for instance, in rock interface analysis in geomechanics and describes a nonmonotone friction law (Panagiotopoulos (1985,1993)).

Remark 2

The above approach applies to a class of problems in mechanics and engineering where the constitutive laws are expressed by means of sub-differential relations. As a model relation we consider the law

$$\boldsymbol{\sigma} \in \boldsymbol{\partial w}(\boldsymbol{\varepsilon}) \text{ in } \boldsymbol{\Omega}$$

between the stress tensor $\boldsymbol{\sigma}$ and the strain tensor $\boldsymbol{\varepsilon}$. Here $\boldsymbol{\partial w}$ denotes the Clarke subdifferential of a function $\boldsymbol{w}\colon \mathbb{R}^{\mathbf{6}} \longrightarrow \mathbb{R}$.

If in the destruction support problem the elastic body is not longer linear, then we replace the Hooke law by the nonlinear multivalued relation:

$$\boldsymbol{w}^{\mathbf{0}}(\boldsymbol{\varepsilon}(\boldsymbol{u}); \boldsymbol{\varepsilon}(\boldsymbol{v})) \geq \sum_{i,j=1}^{n} \boldsymbol{\sigma}_{ij}(\boldsymbol{u})\boldsymbol{\varepsilon}_{ij}(\boldsymbol{v}) \text{ for all } \boldsymbol{v} \in \boldsymbol{V}.$$

Proceeding as before, we obtain the following hemivariational inequality: find $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$\int_\Omega w^0(\varepsilon(u); \varepsilon(v))\, dx + \int_{\Gamma_C} j^0(u_\nu; v_\nu)\, d\sigma \geq \langle l, v \rangle$$

for all $\boldsymbol{v} \in \boldsymbol{V}$.

Two particular cases:

1^0 If $w(\varepsilon) = \frac{1}{2}(C\varepsilon, \varepsilon)_{\mathbb{R}^6} = \frac{1}{2}\sum_{ijkh} c_{ijkh}\varepsilon_{ij}\varepsilon_{hk}$ with symmetric, coercive tensor $C = \{c_{ijkh}\}$, $i, j, k, h = 1, 2, 3$, then $\partial w = \nabla w$, hence we arrive to the Hooke law $\sigma = C\varepsilon$.

2^0 If the function w is continuously differentiable, then we get $\sigma_{ij} = \dfrac{\partial w(\varepsilon)}{\partial \varepsilon_{ij}}$ which corresponds to a nonlinear elastic material.

Elliptic hemivariational inequalities

Consider the following elliptic hemivariational inequality:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ \langle Au, v\rangle_{V^* \times V} + \displaystyle\int_\Omega j^0(u; v)\, dx \geq \langle f, v\rangle_{V^* \times V} \; \forall v \in V. \end{cases}$$

Results on the existence of solutions:

- Existence via regularization procedure combined with the Galerkin method: Rauch (1977), Panagiotopoulos (1981,1993)
- Existence of solutions via the deformation lemma: Chang (1981) by introducing Palais-Smale condition for a locally Lipschitz functions
- Existence of solutions via the theory of pseudomonotone operators: exploiting Browder and Hess (1972), Zeidler (1990), Naniewicz (1989,1992), Naniewicz and Panagiotopoulos (1995)

Hemivariational inequality versus operator inclusion

We associate with the hemivariational inequality: *find* $\boldsymbol{u} \in \boldsymbol{V}$ *such that*

$$\langle \boldsymbol{Au}, \boldsymbol{v} \rangle + \int_{\Gamma_C} j^0(\gamma \boldsymbol{u}; \gamma \boldsymbol{v}) \, d\Gamma \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}$$

the following operator inclusion: *find* $\boldsymbol{u} \in \boldsymbol{V}$ *such that*

$$\boldsymbol{Au} + \gamma^* \partial \boldsymbol{J}(\gamma \boldsymbol{u}) \ni \boldsymbol{f},$$

where

$\gamma \colon \boldsymbol{V} \longrightarrow \boldsymbol{L}^2(\Gamma_C, \mathbb{R}^d)$ is the trace operator, and
$\boldsymbol{J} \colon \boldsymbol{L}^2(\Gamma_C, \mathbb{R}^d) \longrightarrow \mathbb{R}$ is the functional of the form

$$\boldsymbol{J}(\boldsymbol{z}) = \int_{\Gamma_C} j(\boldsymbol{z}(\boldsymbol{x})) \, d\Gamma \quad \text{for } \boldsymbol{z} \in \boldsymbol{L}^2(\Gamma_C, \mathbb{R}^d).$$

Remark: The multivalued term

$$\boldsymbol{F}(\boldsymbol{u}) = \gamma^* \partial \boldsymbol{J}(\gamma \boldsymbol{u})$$

does not have values in $\boldsymbol{H}$.

Static hemivariational inequalities

The following hemivariational inequalities are the weak formulations of static and quasistatic contact problems: *find* $\boldsymbol{u} \in \boldsymbol{V}$ *such that*

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle + \int_{\Gamma_C} j^0(\gamma \boldsymbol{u}, \gamma \boldsymbol{u}; \gamma \boldsymbol{v})\, d\Gamma \geq \langle \boldsymbol{f}, \boldsymbol{v}\rangle \ \text{ for all } \boldsymbol{v} \in \boldsymbol{V}.$$

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle + \int_{\Omega} j^0(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{v})\, d\Gamma \geq \langle \boldsymbol{f}, \boldsymbol{v}\rangle \ \text{ for all } \boldsymbol{v} \in \boldsymbol{V}.$$

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle + \int_{\Gamma_C} \sum_{i=1}^{k} h_i(\gamma \boldsymbol{u})\, j_i^0(\gamma \boldsymbol{u}; \gamma \boldsymbol{v})\, d\Gamma \geq \langle \boldsymbol{f}, \boldsymbol{v}\rangle \ \text{ for all } \boldsymbol{v} \in \boldsymbol{V}.$$

find $\boldsymbol{u} \in \mathcal{V} = L^2(0, T; V)$ *such that*

$$\langle A(t, \boldsymbol{u}(t)), \boldsymbol{v}\rangle \ + \ \langle \int_0^t C(t-s)\boldsymbol{u}(s)\, ds, \boldsymbol{v}\rangle + \\ + \int_{\Gamma_C} j^0(t, \gamma \boldsymbol{u}(t); \gamma \boldsymbol{v})\, d\Gamma \geq \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle$$

for all $\boldsymbol{v} \in \boldsymbol{V}$ and a.e. $t \in (0, T)$.

Three kinds of hemivariational inequalities

(1) An elastic frictional problem

Problem 1. Find a displacement field $\boldsymbol{u}\colon \Omega \longrightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}\colon \Omega \longrightarrow \mathbb{S}^d$ such that

$$\begin{cases} \mathrm{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \ \text{ in } \ \Omega \\ \boldsymbol{\sigma} = \mathcal{F}\varepsilon(\boldsymbol{u}) \ \text{ in } \ \Omega \\ \boldsymbol{u} = \boldsymbol{0} \ \text{ on } \ \Gamma_D \\ \boldsymbol{\sigma\nu} = \boldsymbol{f}_N \ \text{ on } \ \Gamma_N \\ -\sigma_\nu \in \partial j_\nu(u_\nu - g_0) \ \text{ on } \ \Gamma_C \\ -\boldsymbol{\sigma}_\tau \in h_\tau(u_\nu - g_0)\,\partial j_\tau(\boldsymbol{u}_\tau) \ \text{ on } \ \Gamma_C. \end{cases}$$

(1) An elastic frictional problem

Variational formulation of Problem 1 reads as follows.

Find a displacement field $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$\langle \mathcal{F}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v})\rangle_{\mathcal{H}} + \\ + \int_{\Gamma_C} \left(j_\nu^0(u_\nu - g_0; v_\nu) + h_\tau(u_\nu - g_0)\, j_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau) \right) d\Gamma \geq \\ \geq \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^* \times V} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}.$$

(2) A viscoelastic frictional problem

Problem 2. Find a displacement field $\boldsymbol{u}\colon Q = \Omega \times (0, T) \longrightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}\colon Q \longrightarrow \mathbb{S}^d$ such that

$$\begin{cases} \mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \mathbf{0} \ \text{ in } \ Q \\ \boldsymbol{\sigma}(t) = \mathcal{B}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \displaystyle\int_0^t \mathcal{C}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds \ \text{ in } \ Q \\ \boldsymbol{u}(t) = \mathbf{0} \ \text{ on } \ \Sigma_D \\ \boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_N(t) \ \text{ on } \ \Sigma_N \\ -\sigma_\nu(t) \in \partial j_\nu(t, u_\nu(t)) \ \text{ on } \ \Sigma_C \\ -\boldsymbol{\sigma}_\tau(t) \in \partial j_\tau(t, \boldsymbol{u}_\tau(t)) \ \text{ on } \ \Sigma_C. \end{cases}$$

(2) A viscoelastic frictional problem

Variational formulation of Problem 2 reads as follows.

Find a displacement field $u\colon (0,T) \longrightarrow V$ such that $u \in L^2(0,T;V)$ and

$$\langle \mathcal{B}(t,\varepsilon(u(t))) + \int_0^t \mathcal{C}(t-s,\varepsilon(u(s)))\,ds, \varepsilon(v)\rangle_{\mathcal{H}} +$$

$$+ \int_{\Gamma_C} (j_\nu^0(t,u_\nu(t);v_\nu) + j_\tau^0(t,u_\tau(t);v_\tau))\,d\Gamma \geq$$

$$\geq \langle f(t), v\rangle_{V^*\times V} \quad \text{for all } v \in V, \text{ a.e. } t \in (0,T).$$

(3) An electro-elastic frictional problem

Find a displacement field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^d$, an electric potential $\varphi\colon \Omega \to \mathbb{R}$ and an electric displacement field $\boldsymbol{D}\colon \Omega \to \mathbb{R}^d$ such that

$$
\begin{cases}
\mathrm{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \ \text{ in } \ \Omega \\
\mathrm{div}\boldsymbol{D} - q_0 = 0 \ \text{ in } \ \Omega \\
\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{P}^\top \boldsymbol{E}(\varphi) \ \text{ in } \ \Omega \\
\boldsymbol{D} = \mathcal{P}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta \boldsymbol{E}(\varphi) \ \text{ in } \ \Omega \\
\boldsymbol{u} = \boldsymbol{0} \ \text{ on } \ \Gamma_D \\
\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \ \text{ on } \ \Gamma_N \\
\varphi = 0 \ \text{ on } \ \Gamma_a \\
\boldsymbol{D}\cdot\boldsymbol{\nu} = q_b \ \text{ on } \ \Gamma_b \\
-\sigma_\nu \in h_\nu(\varphi - \varphi_0)\, \partial j_\nu(u_\nu - g_0) \ \text{ on } \ \Gamma_C \\
-\boldsymbol{\sigma}_\tau \in h_\tau(\varphi - \varphi_0, u_\nu - g_0)\, \partial j_\tau(\boldsymbol{u}_\tau) \ \text{ on } \ \Gamma_C \\
\boldsymbol{D}\cdot\boldsymbol{\nu} \in h_e(u_\nu - g_0)\, \partial j_e(\varphi - \varphi_0) \ \text{ on } \ \Gamma_C
\end{cases}
$$

(3) An electro-elastic frictional problem

Variational formulation of Problem 3 reads as follows.

Find a displacement field $\boldsymbol{u} \in \boldsymbol{V}$ and an electric potential $\varphi \in \Phi$ such that

$$\langle \mathcal{F}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v})\rangle_{\mathcal{H}} + \langle \mathcal{P}^{\top}\nabla\varphi, \varepsilon(\boldsymbol{v})\rangle_{\mathcal{H}} +$$

$$+ \int_{\Gamma_C} (h_\nu(\varphi - \varphi_0)\, j_\nu^0(u_\nu - g_0; v_\nu) +$$

$$+ h_\tau(\varphi - \varphi_0, u_\nu - g_0)\, j_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau))\, d\Gamma \geq$$

$$\geq \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^* \times V} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V},$$

$$\langle \beta\nabla\varphi, \nabla\psi\rangle_H - \langle \mathcal{P}\varepsilon(\boldsymbol{u}), \nabla\psi\rangle_H +$$

$$+ \int_{\Gamma_C} h_e(u_\nu - g_0)\, j_e^0(\varphi - \varphi_0; \psi)\, d\Gamma \geq \langle q, \psi\rangle_{\Phi^* \times \Phi} \quad \text{for all } \psi \in \Phi,$$

where $\Phi = \{\, \varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \Gamma_a \,\}$

Acknowledgements

Research is supported in part by

- the Marie Curie International Research Staff Exchange Scheme Fellowship within the 7th European Community Framework Programme under Grant Agreement No. 295118, Project People "Contact" (2012–2016)
- the International Project co-financed by the Ministry of Science and Higher Education of Republic of Poland under grant no. W111/7.PR/2012 (2012–2016)
- the National Science Center of Poland under Maestro Advanced Project no. DEC-2012/06/A/ST1/00262 (2013–2018).

The author greatly appreciates the hospitality and a support given in June 2014 by Department of Mathematical Analysis and Applications of Mathematics Faculty of Science, Palacký University Olomouc

Thank you !

stanislaw.migorski@uj.edu.pl