Canonical Duality and Triality:
Unified Understanding and General Solutions to Complex Systems

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ABSTRACT: Duality is a beautiful, fundamental, and inspiring concept that underlies all natural phenomena. According to Sir M.F. Atiyah, duality in mathematics is not a theorem, but a "principle". Therefore, duality gap is not allowed in mathematical physics. For convex systems, the duality theory has been well-developed with extensive applications in mathematical economics, dynamical systems, nonlinear analysis, global optimization, control theory, management and decision science, numerical methods and scientific computation. However, in nonconvex systems, the classical Lagrange multiplier method and Fenchel-Moreau-Rockafellar dualities have been misused for solving nonconvex/discrete problems, and produced different duality gaps. It turns out that many nonconvex and discrete optimization problems are considered to be NP-hard.

The canonical duality-triality is a breakthrough methodological theory, which can be used not only to model complicated phenomena within a unified framework, but also for solving a wide class of nonconvex/nonsmooth/discrete problems in different fields. The associated triality theory reveals an interesting multi-scale duality pattern in complex systems, which can be used to identify both global and local extrema and to design efficient algorithms for solving challenging problems. In this special lecture series, the speaker will present an introduction to the canonical duality-triality theory with a purpose toward a unified understanding complex systems and challenging problems in mathematical physics, continuum mechanics, nonconvex analysis, global optimization, and computational science.

Lecture I: Symmetry and Lagrangian Duality in Geometrically Linear Systems

Beginning with the oriental philosophy and the most simple quadratic optimization problem, the speaker will present a unified mathematical modelling in convex systems. He will show that by using the classical Legendre transformation, the mono-duality in static systems and bi-duality in dynamic systems can be naturally developed. The well-known least action principle is actually a misnomer in convex Hamiltonian systems. Some basic concepts, such as objectivity, Lagrangian, equilibrium equation, geometrical nonlinearity will be discussed along with the common misunderstandings between physics and mathematical programming. Also the geometrical and constitutive constraints will be classified and a bi-complementarity problem will be presented for unified understanding general variational inequality and constrained optimization problems.

Lecture II: Symmetry Breaking and Unified Modelling in Geometrically Nonlinear Systems

By using some interesting experiments in our daily life, the speaker will explain why the symmetry breaking leads to phenomena and what is the fundamental reason that leads to difficulties, such as bifurcation, big-bang theory, chaos in nonlinear dynamics, NP-hard problems in computer science, and the paradox of Buridan's donkey in decision making. Based on oriental yin-yang philosophy and some fundamental principles in systems theory, he will show how nonlinear phenomena can be modelled as a unified nonconvex optimization problem and the canonical duality theory can be naturally, precisely developed, why the well-known nonlinear von Karman plate is actually a linear model in one-dimensional space, and how the Gao beam model was proposed. Using a simple nonconvex variational problem of Ericksen's bar in phase transitions, the speaker will demonstrate that a class of nonlinear differential equations can be transformed to certain algebraic equations and can be solved completely to obtain all possible solutions. A movie will show that both global and local minimal solutions are usually nonsmooth and can’t be captured by any Newton-type numerical approaches, which is one of reasons for NP-hardness.

Lecture III: Canonical Duality and Unified Solution to Nonconvex Variational/Boundary Value Problems

In this lecture, the speaker will first review some well-known complementary variational principles and arguments in nonlinear elasticity and how these principles are unified by the canonical duality theory. Then he will show how the general nonlinear/nonconex partial differential equation in large deformation problems can be converted into an algebraic equation in dual space, which can, in principle, be solved to obtain a complete set of stress solutions. Therefore, a general analytical solution form of the deformation is obtained subjected to a compatibility condition. For 3-D St Venant-Kirchhoff material, the problem could have at most 27 stress solutions. He will show that the Gao-Strang gap function can be used to identify both global and local extrema, while the popular (poly-, quasi-, and rank-one) convexities provide only local minimal criteria, the Legendre-Hadamard condition does not guarantee uniqueness of solutions. The so-called elliptic equation is indeed a misnomer in nonconvex variational analysis.

Lecture IV: Unified Solution to NP-Hard Problems in Global Optimization and Computer Science

In this lecture, the speaker will illustrate that how the canonical duality-triality theory can be used for solving a large class of NP-hard problems in global optimization. By finite element method, connections and challenges in computational mechanics, network optimization, information technology, operations research and systems engineering will be discussed. Combining the canonical duality-triality theory with nonlinear perturbation technique, a powerful canonical dual finite element method and algorithm will be presented, which can be used for solving not only the most well-known problems in continuum physics, such as Landau-Ginzburg, Cahn-Hilliard equations and general nonlinear dynamical systems, but also a large class of so-called “NP-hard” problems in combinatorial optimization and computer science. Applications will be illustrated by complete numerical solutions to post-buckling of a large deformed beam and a sensor network localization problem. Concluding remarks and some open problems will be addressed.

This lecture series will bring some fundamental new insights into modern mechanics, complex systems, and computational science.

Mathematical Experiment:

For a given $f \in \{-1, 1\}$, please plot the nonconvex function $P(x) = ½\,(½\,x^2 - 1)^2 - f\,x$ and its canonical dual $P^d(x) = -½\,x^{-1} f^2 - ½\,x^2 - x$ in the same frame to see the magic triality, why symmetry $(f=0)$ leads to NP-hardness, and the perturbation ($f \neq 0$) for finding global minimizer.

References:

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[2] Gao, D.Y.and Ogden, R.W. (2008) Multiple solutions to non-convex variational problems with implications for phase transitions and numerical computation, *Q. J. Mech. Appl. Math.* . 61 (4), 497-522

[3] Gao, D.Y. and Sherali, H. (2009). Canonical duality: Connection between nonconvex mechanics and global optimization, in *Advances in Appl. Mathematics and Global Optimization,* 249-316, Springer.

[4] Cai, K, Gao, D. , Qin, QH (2013). Post-buckling Solutions of Hyper-elastic Beam by Canonical Dual Finite Element Method, *Mathematics and Mechanics of Solids*, http://arxiv.org/abs/1302.4136

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Professor David Y. Gao is an internationally renowned scholar for his research and professional activities in the fields of applied mathematics, theoretical and engineering mechanics, global optimization, industrial and systems engineering. He received his PhD from Tsinghua University. Since then, he has held research and teaching positions in different institutes including MIT (Math and Ocean Engineering), Yale (Mechanical Engineering), Harvard (Math), University Hong Kong (Civil Engineering), the University of Michigan (Math and Applied Mechanics), and Virginia Tech (Math/Mech/Industrial and Systems Engineering). He moved to Australia in 2010 for his current position as the Alexander Rubinov Chair Professor in School of Science, Information Technology and Engineering at the Federation University. He is also an Adjunct Professor in the Research School of Engineering at ANU.

Professor Gao is the author of about 14 monograph/handbook/special volumes and more than 150 research papers on mathematical modeling, modern mechanics, applied analysis, nonlinear PDEs, nonconvex/nonsmooth variational methods, large deformation structural theory (beams, plates, and shells), complex dynamical systems, bifurcation theory, phase transitions in solids, global optimization and control, operations research, information theory, decision science, numerical methods and computational science with extensive applications in physics, biology, engineering, and general systems.

Professor Gao is an editor-in-chief for three book series including *Advances in Mechanics and Mathematics* by Springer. He is also an associate editor of about eight international journals. Since 2000, Professor Gao has delivered over 30 keynote/plenary/invited lectures at international conferences and more than 60 colloquium talks at different universities and institutions. As the chair and co-chair, he has organized successfully about 10 world congress/conferences. Currently, he is serving as the Secretary-General and Vice President of the International Society of Global Optimization.

Web page:

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