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Analysis of Singular BVPs in ODEs

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General systems – BVPs

Vector case – variable coefficient matrix

Existence and uniqueness of continuous solutions

General systems – BVPs

General systems

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$X_+ \ldots$ invariant subspace of M associated with eigenvalues with positive real parts,

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Assumptions:
A1. IVP: For all λ either $\sigma < 0$ or $\lambda = 0$.

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Assumptions:
A1. IVP: For all λ either $\sigma < 0$ or $\lambda = 0$.

A2. TVP: For all λ either $\sigma > 0$ or $\lambda = 0$. For $\lambda = 0$ the associated invariant subspace is the eigenspace of M.

General systems - IVPs

Consider the system

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0, 1].$$

Lemma: Let A1 hold and $z \in C$ be a solution of the above system. Then

$$Qz(0) = 0, \quad Rz(0) = R\gamma \Leftrightarrow Mz(0) = 0.$$

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$$Qz(0) = 0, \quad Rz(0) = R\gamma \Leftrightarrow Mz(0) = 0.$$

This result means that the requirement $z \in C[0, 1]$ is equivalent to $\operatorname{rank}(Q) = n - \operatorname{rank}(R) = n - r$ homogeneous initial conditions z must satisfy.

General systems - IVPs

Consider the IVP

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad Qz(0) = 0, \quad B_0 z(0) = \beta.$$

General systems - IVPs

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$$z'(t) = \frac{M}{t} z(t) + f(t), \quad Qz(0) = 0, \quad B_0 z(0) = \beta.$$

Lemma: Let A1 hold and let the $r \times r$ matrix $B_0 \tilde{R}$ be nonsingular. Then for every $f \in C^p[0,1]$, $p \geq 0$, and any vector $\beta \in \mathbb{R}^r$, there is a unique solution $z \in C^{p+1}[0,1]$ of the above IVP and it has the form

$$z(t) = \tilde{R}(B_0 \tilde{R})^{-1} \beta + t \int_0^1 s^{-M} f(st)\, ds.$$

General systems - TVPs

Consider the system

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0, 1].$$

Lemma: Let A2 hold and let z be a solution of the above system. Then $z \in C[0, 1]$ and

$$Sz(0) = 0.$$

General systems - TVPs

Consider the system

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0, 1].$$

Lemma: Let A2 hold and let z be a solution of the above system. Then $z \in C[0, 1]$ and

$$Sz(0) = 0.$$

This result means that the smoothness requirement $z \in C[0, 1]$ is satisfied by any solution.

General systems - TVPs

Consider the TVP

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad B_1 z(1) = \beta.$$

General systems - TVPs

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$$z'(t) = \frac{M}{t} z(t) + f(t), \quad B_1 z(1) = \beta.$$

Lemma: Let A2 hold and let the $n \times n$ matrix B_1 be nonsingular. Then for every $f \in C^p[0,1],\ p \geq 0$, and any vector $\beta \in \mathbb{R}^n$, there is a unique solution $z \in C[0,1] \cap C^{p+1}(0,1]$ of the above TVP. This solution is given by

$$z(t) = t^M B_1^{-1} \beta + t^M \int_1^t s^{-M} f(s)\, ds.$$

General systems - BVPs

Consider the BVP

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad Qz(0) = 0, \quad Sz(1) = S\gamma, \quad Rz(0) = R\gamma.$$

General systems - BVPs

Consider the BVP

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad Qz(0) = 0, \quad Sz(1) = S\gamma, \quad Rz(0) = R\gamma.$$

Lemma: For every $f \in C[0,1]$ and every $\gamma \in \mathbb{R}^n$ there exists a unique solution $z \in C[0,1]$ of the above BVP. This solution has the form (T1.20-T1.21)

$$z(t) = t^M (S + R)\gamma + (Kf)(t) = t^M P\gamma + (Kf)(t),$$

where $K : C[0,1] \to C[0,1]$,

$$(Kf)(t) = tQ\int_0^1 s^{-M} f(ts)ds + t^M S\int_1^t s^{-M} f(s)ds + tR\int_0^1 s^{-M} f(ts)ds.$$

General systems - Example1

Solve

$$y'(t) = \frac{M}{t} y(t) + f(t), \quad t \in (0,1], \quad B_0 y(1) = \beta, \;\; B_0 \in \mathbb{R}^{2\times 3},$$

General systems - Example1

Solve

$$y'(t) = \frac{M}{t} y(t) + f(t), \quad t \in (0, 1], \quad B_0 y(1) = \beta, \;\; B_0 \in \mathbb{R}^{2\times 3},$$

with

$$M := \begin{pmatrix} 3 & 3 & -12 \\ -32 & -32 & 124 \\ -8 & -8 & 31 \end{pmatrix}, \quad f(t) := 20 \sin(5t) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

General systems - Example1

Solve

$$y'(t) = \frac{M}{t} y(t) + f(t), \quad t \in (0,1], \quad B_0 y(1) = \beta, \; B_0 \in \mathbb{R}^{2\times 3},$$

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$$M := \begin{pmatrix} 3 & 3 & -12 \\ -32 & -32 & 124 \\ -8 & -8 & 31 \end{pmatrix}, \quad f(t) := 20\sin(5t) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

and

$$B_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; \beta = \begin{pmatrix} 4 - 4\cos(5) + 1 \\ -4 + 4\cos(5) - 1 \end{pmatrix}.$$

General systems - Example2

Calculate eigenvalues and eigenvectors of M:

$$\lambda_1 = 0,\ \lambda_2 = -1,\ \lambda_3 = 3, \quad v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix},\ v_3 = \begin{pmatrix} -1 \\ 8 \\ 2 \end{pmatrix}.$$

General systems - Example2

Calculate eigenvalues and eigenvectors of M:

$$\lambda_1 = 0,\ \lambda_2 = -1,\ \lambda_3 = 3,\quad v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix},\ v_3 = \begin{pmatrix} -1 \\ 8 \\ 2 \end{pmatrix}.$$

Therefore, $M = EJE^{-1}$ with

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$E = \begin{pmatrix} v_1, & v_2, & v_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 4 & 8 \\ 0 & 1 & 2 \end{pmatrix}, \quad E^{-1} = \begin{pmatrix} 0 & 1 & -4 \\ 2 & 2 & -7 \\ -1 & -1 & 4 \end{pmatrix}.$$

General systems - Example3

Construct the projections R, Q and S. It holds

$$Rv_1 = v_1, \quad Rv_2 = 0, \quad Rv_3 = 0,$$
$$Qv_1 = 0, \quad Qv_2 = v_2, \quad Qv_3 = 0,$$
$$Sv_1 = 0, \quad Sv_2 = 0, \quad Sv_3 = v_3.$$

General systems - Example3

Construct the projections R, Q and S. It holds

$$Rv_1 = v_1, \quad Rv_2 = 0, \quad Rv_3 = 0,$$
$$Qv_1 = 0, \quad Qv_2 = v_2, \quad Qv_3 = 0,$$
$$Sv_1 = 0, \quad Sv_2 = 0, \quad Sv_3 = v_3.$$

Therefore,

$$R = \begin{pmatrix} 0 & -1 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \; Q = \begin{pmatrix} 0 & 0 & 0 \\ 8 & 8 & -28 \\ 2 & 2 & -7 \end{pmatrix}, \; S = \begin{pmatrix} 1 & 1 & -4 \\ -8 & -8 & 32 \\ -2 & -2 & 8 \end{pmatrix}.$$

General systems - Example4

Construct the solution according to,

$$z(t) = t^M P\gamma + (Kf)(t),$$

where

$$(Kf)(t) = tQ \int_0^1 s^{-M} f(ts) \mathrm{d}s$$

$$+ t^M S \int_1^t s^{-M} f(s) \mathrm{d}s$$

$$+ tR \int_0^1 s^{-M} f(ts) \mathrm{d}s.$$

General systems - Example5

General solution of the homogeneous problem:

$$t^M = t^{EJE^{-1}} = Et^J E^{-1} = Et^{\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}} E^{-1} \Rightarrow$$

$$t^M = E \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{t} & 0 \\ 0 & 0 & t^3 \end{pmatrix} E^{-1},\ t^{-M} = E \begin{pmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & \frac{1}{t^3} \end{pmatrix} E^{-1}.$$

General systems - Example6

General solution of the homogeneous problem:

$$\Phi(t) = t^M P = \underbrace{E \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{t} & 0 \\ 0 & 0 & t^3 \end{pmatrix} E^{-1}}_{t^M} \left(\underbrace{\begin{pmatrix} 0 & -1 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}}_{R} + \underbrace{\begin{pmatrix} 1 & 1 & -4 \\ -8 & -8 & 32 \\ -2 & -2 & 8 \end{pmatrix}}_{S} \right) =$$

$$\begin{pmatrix} t^3 & -1 + t^3 & 4 - 4t^3 \\ \frac{8}{t} - 8t^3 & 1 + \frac{8}{t} - 8t^3 & -4 - \frac{28}{t} + 32t^3 \\ \frac{2}{t} - 2t^3 & \frac{2}{t} - 2t^3 & -\frac{7}{t} + 8t^3 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -8 & -7 & 28 \\ -2 & -2 & 8 \end{pmatrix}}_{P,\ \text{rank}(P)=2} =$$

$$\begin{pmatrix} t^3 & -1 + t^3 & 4 - 4t^3 \\ -8t^3 & 1 - 8t^3 & -4 + 32t^3 \\ -2t^3 & -2t^3 & 8t^3 \end{pmatrix} \Rightarrow \Phi(1) = t^M P|_{t=1} = P.$$

General systems - Example7

Particular solution of the inhomogeneous problem – contribution Q:

$$tQ \int_0^1 s^{-M} f(ts) \mathrm{d}s =$$

$$tQ \int_0^1 E \begin{pmatrix} 0 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \frac{1}{s^3} \end{pmatrix} E^{-1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} 20 \sin(5ts) \mathrm{d}s = 0,$$

since

$$E s^{-J} E^{-1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdots \text{ eigenvector of } \lambda_1 = 0.$$

General systems - Example8

Particular solution of the inhomogeneous problem – contribution S:

$$t^M S \int_1^t s^{-M} f(s)\mathrm{d}s =$$

$$t^M S \int_1^t E \begin{pmatrix} 0 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \frac{1}{s^3} \end{pmatrix} E^{-1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} 20 \sin(5s)\mathrm{d}s = 0,$$

since

$$E s^{-J} E^{-1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdots \text{ eigenvector of } \lambda_1 = 0.$$

General systems - Example9

Particular solution of the inhomogeneous problem – contribution R:

$$tR\int_0^1 s^{-M} f(ts)\mathrm{d}s =$$

$$tR\int_0^1 E\begin{pmatrix} 0 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \frac{1}{s^3} \end{pmatrix} E^{-1}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} 20\sin(5ts)\mathrm{d}s = \begin{pmatrix} 4-4\cos(5t) \\ -4+4\cos(5t) \\ 0 \end{pmatrix}$$

since

$$REs^{-J}E^{-1}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdots \text{ eigenvector of } \lambda_1 = 0.$$

General systems - Example10

General solution of the inhomogeneous problem:

$$z(t) = t^M P(P\gamma) + (Kf)(t) = \begin{pmatrix} t^3 & -1+t^3 & 4-4t^3 \\ -8t^3 & 1-8t^3 & -4+32t^3 \\ -2t^3 & -2t^3 & 8t^3 \end{pmatrix} (P\gamma) + \begin{pmatrix} 4-4\cos(5t) \\ -4+4\cos(5t) \\ 0 \end{pmatrix},$$

where $\gamma \in \mathbb{R}^3$. Note that

$$P\gamma = \begin{pmatrix} 1 & 0 & 0 \\ -8 & -7 & 28 \\ -2 & -2 & 8 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -8 & -7 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \tilde{P}\eta,$$

where $\eta_1 = \gamma_1, \;\; \eta_2 = \gamma_2 - 4\gamma_3.$

General systems - Example11

Boundary conditions:

$$z(t) = \underbrace{\begin{pmatrix} t^3 & -1+t^3 & 4-4t^3 \\ -8t^3 & 1-8t^3 & -4+32t^3 \\ -2t^3 & -2t^3 & 8t^3 \end{pmatrix}}_{t^M P} \underbrace{\begin{pmatrix} 1 & 0 \\ -8 & -7 \\ -2 & -2 \end{pmatrix}}_{\tilde{P}} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} 4-4\cos(5t) \\ -4+4\cos(5t) \\ 0 \end{pmatrix},$$

where $\eta \in \mathbb{R}^2$. Then, it follows from $B_0 z(1) = \beta$,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -8 & -7 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4-4\cos(5) \\ -4+4\cos(5) \\ 0 \end{pmatrix} = \beta$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ -8 & -7 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} 4-4\cos(5) \\ -4+4\cos(5) \end{pmatrix} = \begin{pmatrix} 4-4\cos(5)+1 \\ -4+4\cos(5)-1 \end{pmatrix}$$

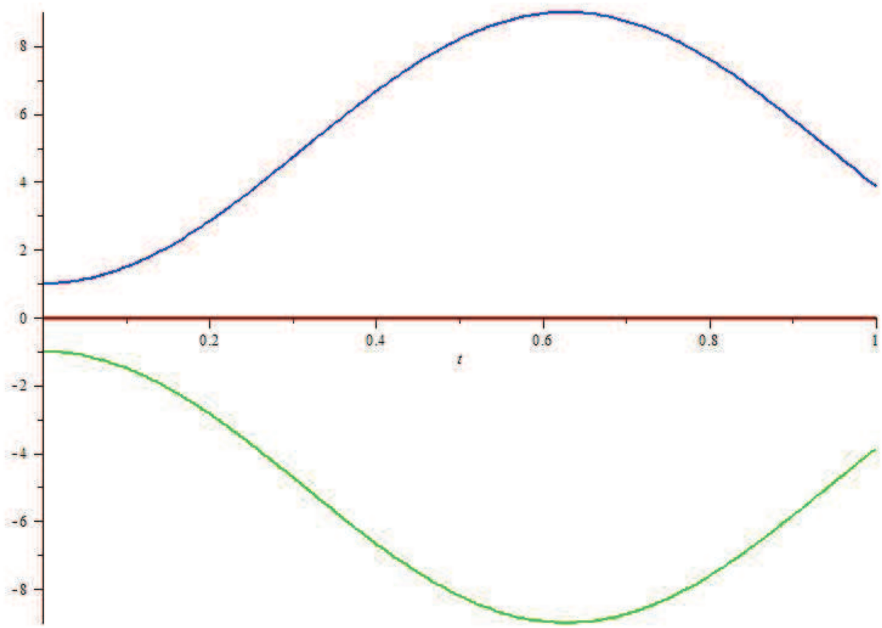
$$\Rightarrow \begin{pmatrix} 1 & 0 \\ -8 & -7 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \eta_1 = 1, \quad \eta_2 = -1.$$

General systems - Example12

The solution reads:

$$z(t) = (t^M P)\tilde{P} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (Kf)(t) = \begin{pmatrix} 5 - 4\cos(5t) \\ -5 + 4\cos(5t) \\ 0 \end{pmatrix}.$$

$z_1(t) = 5 - 4\cos(5t)$, $z_2(t) = -5 + 4\cos(5t)$, $z_3(t) = 0$



General systems - BVPs

Consider the BVP

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad Qz(0) = 0, \;\; B_0 z(0) + B_1 z(1) = \beta.$$

General systems - BVPs

Consider the BVP

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad Qz(0) = 0, \;\; B_0 z(0) + B_1 z(1) = \beta.$$

Lemma: Let $\tilde{P} \in \mathbb{C}^{n \times m}$ be the matrix consisting of the linearly independent columns of P. Then $Y(t) = t^M \tilde{P}$ is the unique continuous $n \times m$ matrix satisfying

$$Y'(t) = \frac{M}{t} Y(t), \quad t \in [0,1], \quad Y(1) = \tilde{P}.$$

Moreover, there exists a unique solution of the BVP iff for the matrices $B_0,\, B_1 \in \mathbb{R}^{m \times n}$ with $m = \text{rank}(P)$, and the right hand side $\beta \in \mathbb{R}^m$, the $m \times m$ matrix

$$B_0 R Y(0) + B_1 \tilde{P}$$

is nonsingular.

Variable coefficient matrix

Recall the BVP for constant coefficient matrix M,

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad Qz(0) = 0, \; Sz(1) = S\gamma, \; Rz(0) = R\gamma$$

with the solution

$$z(t) = t^M (S + R)\gamma + (Kf)(t) = t^M P\gamma + (Kf)(t),$$

where $K : C[0,1] \to C[0,1]$,

$$(Kf)(t) = tQ \int_0^1 s^{-M} f(ts) ds + t^M S \int_1^t s^{-M} f(s) ds + tR \int_0^1 s^{-M} f(ts) ds.$$

Variable coefficient matrix

Consider the variable coefficient matrix $M(t) := M + C(t)$,

$$z'(t) = \frac{M(t)}{t} z(t) + f(t), \quad Qz(0) = 0, \;\; Sz(1) = S\gamma, \;\; Rz(0) = R\gamma.$$

Now, we have

$$z'(t) = \frac{M + tC(t)}{t} z(t) + f(t) = \frac{M}{t} z(t) + \underbrace{C(t)z(t) + f(t)}_{f(t,z(t))}$$

with the solution

$$z(t) = t^M P\gamma + (Kf)(t, z(t)), \;\; K : C[0,1] \to C[0,1],$$

$$(Kf)(t, z(t)) = tQ \int_0^1 s^{-M} (C(ts)z(ts) + f(ts))ds + t^M S \int_1^t s^{-M} (C(s)z(s) + f(s))ds +$$
$$tR \int_0^1 s^{-M} (C(ts)z(ts) + f(ts))ds = (KCz)(t) + (Kf)(t).$$

Banach Fixpoint Theorem

Consider the BVP on $[0,\delta]$, $\delta < 1$,

$$z'(t) = \frac{M(t)}{t} z(t) + f(t), \quad Qz(0) = 0, \;\; Sz(\delta) = S\gamma, \;\; Rz(0) = R\gamma.$$

Banach Fixpoint Theorem

Consider the BVP on $[0, \delta]$, $\delta < 1$,

$$z'(t) = \frac{M(t)}{t} z(t) + f(t), \quad Qz(0) = 0, \;\; Sz(\delta) = S\gamma, \;\; Rz(0) = R\gamma.$$

Study the fixpoint equation $z = \mathcal{K}z$, where

$$\mathcal{K}z(t) := \left(\left(\frac{t}{\delta} \right)^M S + t^M R \right) \gamma + (KCz)(t) + (Kf)(t), \;\; \mathcal{K} : (C[0, \delta], \|\cdot\|_\delta) \to (C[0, \delta], \|\cdot\|_\delta),$$

where

$$(Kf)(t, z(t)) = tQ \int_0^1 s^{-M} (C(ts)z(ts) + f(ts))ds + \left(\frac{t}{\delta} \right)^M S \int_\delta^t \left(\frac{s}{\delta} \right)^{-M} (C(s)z(s) + f(s))ds +$$
$$tR \int_0^1 s^{-M} (C(ts)z(ts) + f(ts))ds.$$

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$$\mathcal{K}z(t) := \left(\left(\frac{t}{\delta} \right)^M S + t^M R \right) \gamma + (KCz)(t) + (Kf)(t), \;\; \mathcal{K} : (C[0, \delta], \|\cdot\|_\delta) \to (C[0, \delta], \|\cdot\|_\delta),$$

where

$$(Kf)(t, z(t)) = tQ \int_0^1 s^{-M} (C(ts)z(ts) + f(ts))ds + \left(\frac{t}{\delta} \right)^M S \int_\delta^t \left(\frac{s}{\delta} \right)^{-M} (C(s)z(s) + f(s))ds + tR \int_0^1 s^{-M} (C(ts)z(ts) + f(ts))ds.$$

We show that $\mathcal{K}$ is a contraction on $(C[0, \delta], \|\cdot\|_\delta)$,

$$\|\mathcal{K}z_1 - \mathcal{K}z_2\|_\delta = \|(KCz_1) - (KCz_2)\|_\delta \le L\|z_1 - z_2\|_\delta, \quad L < 1.$$

Banach Fixpoint Theorem

To this aim we have to estimate three integrals on $[0, \delta]$,

$$Q((KCz_1)(t) - (KCz_2)(t)) = tQ \int_0^1 s^{-M} C(ts)(z_1(ts) - z_2(ts))ds,$$

$$S((KCz_1)(t) - (KCz_2)(t)) = t^M S \int_\delta^t s^{-M} C(s)(z_1(s) - z_2(s)))ds,$$

$$R((KCz_1)(t) - (KCz_2)(t)) = tR \int_0^1 s^{-M} C(ts)(z_1(ts) - z_2(ts))ds.$$

Banach Fixpoint Theorem - Projection Q

First integral:

$$\|Q((KCz_1)-(KCz_2))\|_\delta = \max_{t\in[0,\delta]} \left| t\int_0^1 Qs^{-M}C(ts)(z_1(ts)-z_2(ts))ds \right| \le$$

$$\max_{t\in[0,\delta]} \left\{ t \int_0^1 \left| Qs^{-M}C(st)(z_1(st) - z_2(st)) \right| ds \right\} \le$$

$$\max_{t\in[0,\delta]} \left\{ t \int_0^1 \left| Qs^{-M} \right| ds \cdot \max_{s\in[0,t]} |C(s)| \cdot \max_{s\in[0,t]} |z_1(s) - z_2(s)| \right\} \le$$

$$\underbrace{\delta \cdot \text{const} \cdot \|C\|_\delta}_{<\frac{1}{3},\ \text{for } \delta \text{ sufficiently small}} \ \|z_1 - z_2\|_\delta$$

Banach Fixpoint Theorem - Projection R

Second integral:

$$\|R((KCz_1)-(KCz_2))\|_\delta = \max_{t\in[0,\delta]} \left| t\int_0^1 Rs^{-M}C(ts)(z_1(ts)-z_2(ts))ds \right| \leq$$

$$\max_{t\in[0,\delta]} \left\{ t \int_0^1 \left| Rs^{-M}C(st)(z_1(st) - z_2(st)) \right| ds \right\} \leq$$

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Banach Fixpoint Theorem - Projection S

Third integral:

$$\|S((KCz_1)-(KCz_2))\|_\delta = \max_{t\in[0,\delta]} \left| t^M S \int_\delta^t s^{-M} C(s)(z_1(s)-z_2(s))ds \right| \leq$$

$$\max_{t\in[0,\delta]} \left\{ \int_\delta^t \left| S \left(\frac{t}{s}\right)^M C(s)(z_1(s)-z_2(s)) \right| ds \right\} \leq$$

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We discussed existence, uniqueness and smoothness of solutions $z \in C[0,1]$ of the BVP

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Thank you for your attention!