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Streamlining the Mathematics Studies at the Faculty of Science of Palacky University in Olomouc

Registry number: CZ.1.07/2.2.00/28.0141

Analysis of Singular BVPs in ODEs

Ewa Weinmüller

Analysis and Scientific Computing, Vienna University of Technology

Palacky University, Olomouc, Czech Republic, 25 – 27 November 2013

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Vector case – variable coefficient matrix

Existence and uniqueness of continuous solutions

General systems – BVPs

Bounded solutions: $\lambda = 0$

Case 2: $\lambda = 0$

For $\lambda = 0$, the DE $z'(t) = \frac{\lambda}{t} z(t) + f(t)$ is simply $z'(t) = f(t)$.

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Its general solution reads

$$z(t) = t^{\lambda} d + t^{\lambda} \int_1^t s^{-\lambda} f(s) ds = t^{\lambda} c + t \int_0^1 s^{-\lambda} f(ts) ds,$$

or equivalently

$$z(t) = c + t \int_0^1 f(ts) ds, \quad z'(t) = f(t), \quad z^{(k+1)}(t) = f^{(k)}(t).$$

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or equivalently

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Clearly, $f \in C^p[0,1]$ implies $z \in C^{p+1}[0,1]$.

Summarize the case $\lambda = 0$

Theorem: Consider the following IVP:

$$z'(t) = f(t), \quad z(0) = \beta.$$

Then, for any $f \in C^p[0,1]$ there exist unique solutions $z \in C^{p+1}[0,1]$ of the IVP. This solution satisfies

$$z(t) = \beta + t \int_0^1 f(ts)ds$$

and

$$|z(t)| \leq |\beta| + t||f||, \quad |z'(t)| \leq ||f||.$$

Bounded solutions: $\lambda > 0$

Case 3: $\lambda > 0$

The general solution again reads:

$$z(t) = z_h(t) + z_p(t) = t^\lambda c + t^\lambda \int_1^t s^{-\lambda} f(s) ds.$$

Since (T1.5-T1.6),

$$|z_p(t)| \leq ||f|| \begin{cases} \frac{1}{1-\lambda} t^\lambda, & \lambda < 1, \\ t|\ln t|, & \lambda = 1, \\ \frac{1}{\lambda - 1} t, & \lambda > 1. \end{cases}$$

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$z(t) \in C[0,1]$ for any $f(t) \in C[0,1]$ and $z(0) = 0$ holds.

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The general solution again reads:

$$z(t) = z_h(t) + z_p(t) = t^{\lambda} c + t^{\lambda} \int_1^t s^{-\lambda} f(s) ds.$$

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$z(t) \in C[0,1]$ for any $f(t) \in C[0,1]$ and $z(0) = 0$ holds.

Also for $\lambda > 1$, $z \in C^1[0,1]$ since

$$z'(t) = \lambda t^{\lambda - 1} c + \lambda t^{\lambda - 1} \int_1^t s^{-\lambda} f(s) ds + f(t).$$

Summarize the case $\lambda > 0$

Theorem: Consider the following TVP:

$$z'(t) = \frac{\lambda}{t} z(t) + f(t), \quad t \in (0,1], \quad z(1) = \beta.$$

Then, for any $f \in C^p[0,1]$ there exist a unique solution $z \in C[0,1] \cap C^{p+1}(0,1]$ of the TVP. This solution satisfies

$$z(t) = t^\lambda \beta + t^\lambda \int_1^t s^{-\lambda} f(s) ds,$$

and

$$|z(t)| \le \text{const} \begin{cases} t^\lambda(|\beta| + ||f||), & \lambda < 1, \\ t(|\beta| + |\ln t|||f||), & \lambda = 1, \\ t(|\beta| + ||f||), & \lambda > 1, \end{cases} \quad |z'(t)| \le \text{const} \begin{cases} t^{\lambda-1}(|\beta| + ||f||), & \lambda < 1, \\ (|\beta| + |\ln t|||f||), & \lambda = 1, \\ (|\beta| + ||f||), & \lambda > 1. \end{cases}$$

Moreover, for $\lambda > p+1$, $z \in C^{p+1}[0,1]$.

Vector case - constant coefficient matrix

Consider the IVP

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0, 1], \quad B_0 z(0) = \beta,$$

where $M \in \mathbb{R}^{n \times n}$, $B_0 \in \mathbb{R}^{m \times n}$, and $\beta \in \mathbb{R}^m$, $m \leq n$,

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- The aim: Formulate initial/terminal conditions necessary and sufficient for $z \in C[0,1]$.

General solution of the ODE system

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Decoupling: Let J be the Jordan canonical form of M and E the associated matrix of generalized eigenvectors such that

$$M = EJE^{-1}.$$

Let

$$v(t) = E^{-1} z(t), \quad g(t) := E^{-1} f(t),$$

then

$$v'(t) = \frac{J}{t} v(t) + g(t).$$

General solution of the ODE system

Consider the system

$$v'(t) = \frac{J}{t} v(t) + g(t), \quad t \in (0, 1].$$

Assume $J \in \mathbb{C}^{n \times n}$ to consist of only one box,

$$J = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ 0 & & & \lambda \end{pmatrix}, \quad \lambda = \sigma + \mathrm{i}\rho \in \mathbb{C}.$$

General solution of the ODE system

Lemma: Every solution of $v'(t) = \frac{J}{t} v(t) + g(t)$ has the form

$$v(t) = v_h(t) + v_p(t) = \Phi(t)c + \Phi(t) \int_1^t \Phi^{-1}(\tau) g(\tau)\, d\tau,$$

where $c \in \mathbb{R}^n$ and $\Phi(t) = t^J := \exp(J \ln(t))$ satisfies (T1.7)

$$\Phi'(t) = \frac{J}{t} \Phi(t), \quad \Phi(1) = I, \qquad t \in (0,1],$$

and

$$\Phi(t) = t^J = t^\lambda \begin{pmatrix} 1 & \ln(t) & \frac{\ln(t)^2}{2} & \dots & \frac{\ln(t)^{n-1}}{(n-1)!} \\ 0 & 1 & \ln(t) & \dots & \frac{\ln(t)^{n-2}}{(n-2)!} \\ 0 & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ln(t) \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

Proof.

General solution of the ODE system

For

$$v'(t) = \frac{J}{t} v(t) + g(t), \quad t \in (0, 1]$$

we have

$$v(t) = v_h(t) + v_p(t) = t^J c + t^J \int_1^t \tau^{-J} g(\tau)\, d\tau.$$

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Recall $z(t) = Ev(t)$ and $f(t) = Eg(t)$. Then for

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0,1]$$

we obtain (T1.10-T1.11)

$$z(t) = z_h(t) + z_p(t) = t^M c + t^M \int_1^t s^{-M} f(s)\, ds.$$

Case 1: Eigenvalues with negative reals parts

Lemma: Let $\gamma > 0$ and in J assume for the eigenvalue $\lambda = \sigma + \mathrm{i}\rho$, either $\sigma < 0$ or $\lambda = 0$. Then for

$$u(t) := t^\gamma \int_0^1 s^{-J} s^{\gamma-1} f(st)\, ds, \quad f \in C[0,1],$$

the following estimate holds (T1.8-T1.9):

$$|u(t)| \le \text{const}\, t^\gamma \|f\|_\delta, \quad t \in [0,\delta],$$

where $\|f\|_\delta := \max_{t\in[0,\delta]} |f(t)|$ and $|f(t)| := \max_{1\le i\le n} |f_i(t)|$.

Proof.

Case 1: Eigenvalues with negative reals parts

Lemma: Let all eigenvalues of M have negative real parts. Then for every $f \in C^p[0,1],\ p \geq 0$, there exists a unique solution $z \in C[0,1]$ of $z'(t) = \frac{M}{t} z(t) + f(t)$. This solution has the form (T1.12-T14)

$$z(t) = t \int_0^1 s^{-M} f(st)\, ds,$$

and satisfies $z(0) = 0$ with $z \in C[0,1] \quad \Leftrightarrow \quad z(0) = 0$.

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Moreover, $z \in C^{p+1}[0,1]$ and the following estimates hold (with $\|f\| := \max_{t \in [0,1]} |f(t)|$):

$$|y(t)| \leq \text{const}\, t \|f\|, \qquad |y'(t)| \leq \text{const}\, \|f\|.$$

Summarize: Eigenvalues with negative reals parts

The last lemma implies that in the case where all eigenvalues of M have negative real parts, the initial value problem

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0,1],$$
$$B_0 z(0) = \beta,$$

reduces to

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad t \in (0,1],$$
$$z(0) = 0,$$

and $z \in C^{p+1}[0,1]$ for any $f \in C^p[0,1]$.

Case 2: Eigenvalues $\lambda = 0$

Definitions:

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For simplicity, we select a basis in which M is reduced to J and use this basis to construct the projections.

Case 2: Eigenvalues $\lambda = 0$

Lemma: Let all eigenvalues of M be zero. Then for every $f \in C^p[0,1],\ p \geq 0$, and every $\gamma \in \mathcal{R}(R)$, there exists a solution $z \in C[0,1]$ of $z'(t) = \frac{M}{t} z(t) + f(t)$. This solution satisfies $Mz(0) = 0$ and has the form (T1.15-T1.17)

$$z(t) = \gamma + t \int_0^1 s^{-M} f(st)\, ds.$$

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Let B_0 be a $r \times n$ matrix and assume that the $r \times r$ matrix $B_0 \tilde{R}$ is nonsingular. Then for any $\beta \in \mathbb{R}^r$ there exists a unique solution $z \in C[0,1]$ with $z'(t) = \frac{M}{t} z(t) + f(t)$, $B_0 z(0) = \beta$, and $\gamma = \tilde{R}(B_0 \tilde{R})^{-1} \beta$.

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Moreover, $z \in C^{p+1}[0,1]$ and the following estimates hold:

$$|z(t)| \leq \text{const}\, t \|f\| + |\tilde{R}(B_0 \tilde{R})^{-1} \beta|, \quad |z'(t)| \leq \text{const}\, \|f\|.$$

Summarize: Eigenvalues $\lambda = 0$

The last lemma implies that in the case where all eigenvalues of M are zero, the initial value problem

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad B_0 z(0) = \beta,$$

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$$z'(t) = \frac{M}{t} z(t) + f(t), \quad Mz(0) = 0, \; B_0 z(0) = \beta.$$

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$n - r$ initial conditions $Mz(0) = 0 \;\Leftrightarrow\; z \in C[0,1]$ mean that $Hz(0) = 0$, $z(0) \in \ker M$ or $z(0) = Rz(0)$.

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r additional conditions for the uniqueness of z are given by $B_0 z(0) = \beta$, with $B_0 \in \mathbb{R}^{r \times n}$, $\beta \in \mathbb{R}^r$.

Case 3: Eigenvalues with positive real parts

Lemma: Let $\gamma > 0$ and for the eigenvalues of J, $\lambda = \sigma + \mathrm{i}\rho$, assume $\sigma > 0$. Then for

$$u(t) := \int_t^\delta \left(\frac{t}{\tau}\right)^J \tau^{\gamma-1}\, d\tau, \quad t \in [0, \delta]$$

the following estimates hold:

$$|u(t)| \le \begin{cases} \text{const} \left(\frac{t}{\delta}\right)^\sigma \left(1 + \left|\ln\left(\frac{t}{\delta}\right)\right|^{n-1}\right) \delta^\gamma, & \sigma < \gamma, \\ \text{const}\, t^\sigma \left(1 + \left|\ln\left(\frac{t}{\delta}\right)\right|^{n}\right), & \sigma = \gamma, \\ \text{const}\, t^\gamma, & \sigma > \gamma. \end{cases}$$

Proof.

Case 3: Eigenvalues with positive real parts

Lemma: Let all eigenvalues of M have positive real parts. Then for every $f \in C^p[0,1]$, $p \geq 0$, and every $c \in \mathbb{R}^n$, there exists a solution $z \in C[0,1]$ of $z'(t) = \frac{M}{t} z(t) + f(t)$.
This solution has the form (T1.18-T1.19)

$$z(t) = t^M c + t^M \int_1^t s^{-M} f(s)\, ds.$$

If the matrix B_1 is nonsingular, then there exists a unique solution of

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad B_1 z(1) = \beta,$$

with $c = B_1^{-1} \beta$.

Case 3: Eigenvalues with positive real parts

Lemma: The above unique solution of the TVP satisfies $z \in C[0,1] \cap C^{p+1}(0,1]$ and the following estimates hold:

$$|z(t)| \leq \begin{cases} \text{const}\, t^{\sigma_+}(1 + |\ln(t)|^{n_{\max}-1})(|B_1^{-1}\beta| + \|f\|), & \sigma_+ < 1, \\ \text{const}\, t(1 + |\ln(t)|^{n_{\max}})(|B_1^{-1}\beta| + \|f\|), & \sigma_+ = 1, \\ \text{const}\, t(|B_1^{-1}\beta| + \|f\|), & \sigma_+ > 1, \end{cases}$$

$$|z'(t)| \leq \begin{cases} \text{const}\, t^{\sigma_+ - 1}(1 + |\ln(t)|^{n_{\max}-1})(|B_1^{-1}\beta| + \|f\|), & \sigma_+ < 1, \\ \text{const}\, (1 + |\ln(t)|^{n_{\max}})(|B_1^{-1}\beta| + \|f\|), & \sigma_+ = 1, \\ \text{const}\, (|B_1^{-1}\beta| + \|f\|), & \sigma_+ > 1, \end{cases}$$

$\sigma_+ \ldots$ the smallest of the positive real parts,
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$\sigma_+ \ldots$ the smallest of the positive real parts,
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If $p < \sigma_+ \le p+1$, then $z \in C^p[0,1] \cap C^{p+1}(0,1]$ and if $p+1 < \sigma_+$, then $z \in C^{p+1}[0,1]$.

Summarize: Eigenvalues with positive real parts

In case where all eigenvalues of M have positive real parts, for any $f \in C^p[0,1]$ and a nonsingular matrix $B_1 \in \mathbb{R}^{n\times n}$, the terminal value problem

$$z'(t) = \frac{M}{t} z(t) + f(t), \quad B_1 z(1) = \beta,$$

has a unique solution $z \in C[0,1] \cap C^{p+1}(0,1]$, of the form

$$z(t) = t^M B_1^{-1}\beta + t^M \int_1^t s^{-M} f(s)\, ds.$$

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Is $B_1 \in \mathbb{R}^{n\times n}$ nonsingular and $\beta \in \mathbb{R}^n$, then n terminal conditions given by $B_1 z(1) = \beta$ guarantee the uniqueness of z.

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Is $B_1 \in \mathbb{R}^{n\times n}$ nonsingular and $\beta \in \mathbb{R}^n$, then n terminal conditions given by $B_1 z(1) = \beta$ guarantee the uniqueness of z.

A well-posed IVP does not exist.

If $p < \sigma_+ \le p+1$, then $z \in C^p[0,1] \cap C^{p+1}(0,1]$ and if $p+1 < \sigma_+$, then $z \in C^{p+1}[0,1]$.

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