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Analysis of Singular BVPs in ODEs

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- *Existence and uniqueness of continuous solutions*

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Vector case – variable coefficient matrix

- *Existence and uniqueness of continuous solutions*

General systems – BVPs

Motivation: Complex Ginzburg-Landau equation

$$\mathrm{i}\frac{\partial u}{\partial t} + (1 - \mathrm{i}\varepsilon)\Delta u + (1 + \mathrm{i}\delta)|u|^2 u = 0, \quad t > 0 \;\; x \in \mathbb{R}^3$$

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We are interested in self-similar blow-up solutions

Ansatz

$$u(x,t) = L(\tau) y(\tau)$$

$$\tau = \tau(x,t) \quad \lim_{t \to T} L(\tau(x,t)) = \infty$$

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Nonlinear optics, models of turbulence, superconductivity

Budd, Koch, EW (2006)

Similarity reduction

Find $y : [0, \infty) \to \mathbb{C}$

$$(1-\mathrm{i}\varepsilon)\left(y''(\tau) + \frac{2}{\tau}y'(\tau)\right) - y(\tau) + \mathrm{i}a(\tau y(\tau))' + (1+\mathrm{i}\delta)|y(\tau)|^2 y(\tau) = 0$$

$$y'(0) = 0, \quad \Im y(0) = 0, \quad \lim_{\tau\to\infty} \tau y'(\tau) = 0$$

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Implicit, parameter dependent ODE with singular points and unknown parameter a posed on a semi-infinite interval

Need: Analysis and dependable software (error estimation and grid adaptation)

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Questions:

- Does such a solution exist?
- If so, what about its smoothness?
- Do standard numerical schemes for BVPs stay robust?
- How about controlling mechanisms, error estimates, grid selection strategy?
- Does standard software work, or do we need to write our own?

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- *Provide theory and software for non-standard data!*

  2008 - : Auzinger, Koch, Weinmüller, Amodio, Settanni, Levitina, Budd, März, Rachunkova, Vampolova, Vainikko, ...

Stability of Nonlinear Operator Equations and Their Discretizations

H.B. Keller: Approximation Methods for Nonlinear Problems with application to Two-Point BVPs, *Math. Comp.* **29**, pp. 464-474 (1975)

Nonlinear operator equation,

$$F(x) = 0, \quad F : B_1 \to B_2,$$

were $B_1,\ B_2$ are Banach spaces.

Application,

$$y'(t) - \frac{M}{t} y(t) - f(t, y(t)), \quad t \in (0, 1],$$
$$y \in C[0, 1], \quad b(y(0), y(1)).$$

$y,\ f$ are vector-valued functions of dimension n, M is an $n \times n$ matrix, and b is a vector-valued function of dimension $m \le n$.

Aim: Study the properties of the related discrete problem

$$F_h(x_h) = 0, \quad F_h : B_{1,h} \to B_{2,h},$$

where h is a discretization parameter (step size).

Main result - analytical problem

Let $F(x) = 0, \ \ F : B_1 \to B_2$.

Consider the sphere $S_\rho(u) := \{x : x \in B_1, \|x - u\| \le \rho\}$.

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Definition: *The mapping $F(\cdot)$ is stable on $S_\rho(u)$ iff there exists a constant $K_\rho > 0$ such that*

$$\|v - w\| \leq K_\rho \|F(v) - F(w)\|,$$

for all $v, w \in S_\rho(u)$.

A solution $x = u$ of $F(x) = 0$ is stable iff $F(\cdot)$ is stable on $S_\rho(u)$ for some $\rho > 0$.

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- Stable solution u is unique in $S_\rho(u)$.
- Let F be linear, and $Fv = g$, then

  v depends continuously on g, $\|v\| \leq K_\rho \|g\|$.

Main result - analytical problem

Let $F : B_1 \to B_2$ and let $x = u$ be a solution of $F(x) = 0$.
Consider the linearization of $F(x) = 0$ at u:

$$L(u)y = 0, \;\; L(u) : B_1 \to B_2,$$

where $L(x)$ is the Fréchet derivative of F at point x.
The operator $L(x)$ is a linear operator, $L(x) : B_1 \to B_2$ such that

$$r(x, y) = \frac{\|F(x + y) - [F(x) + L(x)y]\|}{\|y\|} \to 0 \text{ as } \|y\| \to 0.$$

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Definition: *A solution u of $F(x) = 0$ is isolated iff $L(u)$ exists and is nonsingular, that is: $L(u)y = 0 \Leftrightarrow y = 0.$*

The above conditions mean that $L(u)$ is injective. From now on, we assume that $L(u) : \mathcal{D}(L(u)) = B_1 \to \mathcal{R}(L(u)) = B_2$ and so $L(u)$ is bijective and $L^{-1}(u)$ exists and is bounded.

Analytical problem: stable vs. isolated solution

Theorem 1. Let u be a stable solution of $F(x) = 0$. Then, if $L(u)$ exists, then it is nonsingular and consequently u is isolated.

Analytical problem: stable vs. isolated solution

Theorem 1. Let u be a stable solution of $F(x) = 0$. Then, if $L(u)$ exists, then it is nonsingular and consequently u is isolated.

Proof: The proof is indirect. We assume that $L(u)y = 0$ but $\|y\| \neq 0$.
In this case, for all positive scalars $a < \rho/\|y\|$ it follows that $v(a) = u + ay \in S_\rho$.
The fact that $F(\cdot)$ is stable on S_ρ yields

$$\|u - v(a)\| \leq K_\rho \|F(u) - F(v(a))\|.$$

We now set $x := u$, $y := ay$ in $r(x, y)$, and obtain $r(u, ay) = \frac{\|F(v(a)) - [F(u) + L(u)ay]\|}{\|ay\|}$ and therefore, $r(u, ay)\|ay\| = \|(F(v(a)) - F(u)) - L(u)ay\|$. From the triangle inequality,

$$r(u, ay)\|ay\| + \|L(u)ay\| \geq \|(F(v(u)) - F(u))\|,$$

and we conclude that $\|u - v(a)\| \leq K_\rho\{r(u, ay)\|ay\| + \|L(u)ay\|\}$. This is equivalent to $a\|y\| \leq K_\rho r(u, ay)a\|y\|$. If we choose $a > 0$, so small that $K_\rho r(u, ay) < 1$, then we arrive at a contradiction. Thus $\|y\| = 0$. (T0.1)

Analytical problem: stable vs. isolated solution

Theorem 2. Let $L(u)$ be nonsingular for some $u \in B_1$ and have a bounded inverse.

Let $L(x)$ exist and be Lipschitz continous on S_{ρ_0} for a ρ_0:

This means that for some constant $K_L > 0$,

$$\|L(x) - L(y)\| \leq K_L \|x - y\|$$

for all $x, y \in S_{\rho_0}$.

Then, $F(\cdot)$ is stable on S_ρ for $\rho < (K_L \|L^{-1}(u)\|)^{-1}$and the stability constant is

$$K_\rho = \|L^{-1}(u)\| (1 - \rho K_L \|L^{-1}(u)\|)^{-1}.$$

Analytical problem: stable vs. isolated solution

Proof: From the generalized mean value theorem, we see that for any $x, y \in S_\rho$ with $\rho \le \rho_0$,

$$F(x) - F(y) = \widetilde{L}(x,y)(x-y),$$

where

$$\widetilde{L}(x,y) = \int_0^1 L(tx + (1-t)y)dt.$$

Now, we show that $\widetilde{L}$ is nonsingular and the norm of $\widetilde{L}^{-1}(x,y)$ is bounded. Clearly,

$$\widetilde{L}(x,y) = L(u) + (\widetilde{L}(x,y) - L(u))$$

and note that

$$\begin{aligned} \|\widetilde{L}(x,y) - L(u)\| &\le \int_0^1 \|L(tx + (1-t)y) - L(\underbrace{tu + (1-t)u}_{=u}\|dt \\ &\le K_L \int_0^1 \underbrace{\|t(x-u) + (1-t)(y-u)\|}_{\le t\rho + (1-t)\rho} dt \le K_L \rho, \end{aligned}$$

where we used the fact that L is Lipschitz continuous. We now use the so-called Banach Lemma to find an upper bound for $\|\widetilde{L}^{-1}(x,y)\|$.

Analytical problem: stable vs. isolated solution

Recall the Banach Lemma:

Consider $A, B \in \mathbb{K}^{n\times n}$, A nonsingular and $\|A^{-1}B\| < 1$. Then, $A - B$ is also nonsingular and

$$\|(A-B)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}B\|}.$$

Now we apply this lemma as follows:
Let $A = L(u)$ and $B = -\widetilde{L}(x,y) + L(u)$. If ρ is so small that $\rho K_L \|L^{-1}\| < 1$ holds, then the Banach Lemma implies that $A - B = \widetilde{L}(x,y)$ is nonsingular and

$$\|\widetilde{L}^{-1}(x,y)\| \leq \frac{\|L^{-1}(u)\|}{1 - \rho K_L \|L^{-1}(u)\|}.$$

This means that $F(\cdot)$ is stable and the norm of $\widetilde{L}^{-1}$ is the stability constant of F.

Finally we note that if u is an isolated solution of $F(x) = 0$, then $L(u)$ exists and is nonsingular. So if, in addition, $L(x)$ exists and is Lipschitz continuous in some $S_{\rho_0}(u)$, then u is stable. This completes the proof. (T0.2)

Focus: Isolated solution

Let $B_1 := C^1[0,1]$ and $B_2 = C[0,1] \times \mathbb{R}^n$

$$F(y)(t) := \left\{ y'(t) - \left(\frac{M}{t} y(t) + f(t, y(t)) \right) = 0, \; b(y(0), y(1)) = 0. \right.$$

have an isolated solution. This means that the linear problem,

$$L(y)v(t) := \left\{ v'(t) - \frac{A(t)}{t} v(t) = 0, \; B_0 v(0) + B_1 v(1) = 0, \right.$$

where $L(y) : B_1 := C^1[0,1] \to B_2 = C[0,1] \times \mathbb{R}^n$, and

$$A(t) = M + t \frac{\partial f(t, y(t))}{\partial y}, \quad B_0 = \frac{b(y(0), y(1))}{\partial y(0)}, \quad B_1 = \frac{\partial b(y(0), y(1))}{\partial y(1)}$$

has only the trivial solution.

Focus: Unique solution

We derive conditions for the the unique solvability of the singular problem

$$L(y)v(t) := \left\{ v'(t) - \frac{A(t)}{t} v(t) = f(t),\ B_0 v(0) + B_1 v(1) = \beta, \right.$$

where

$$L(y) : B_1 := C^1[0,1] \to B_2 = C[0,1] \times \mathbb{R}^n$$

and

$$A(t) \in C^1[0,1],\ \ B_0, B_1 \in \mathbb{R}^{n \times n},\ \beta \in \mathbb{R}^n.$$

Scalar problem of the 1. order

F.R. de Hoog and R. Weiss: Difference methods for BVPs with a singularity of the first kind,

SIAM J. Num. Anal. **13**, pp. 776-813 (1976)

Consider linear scalar problem

$$z'(t) = \frac{\lambda}{t} z(t) + f(t), \;\; t \in (0,1], \;\; f \in C[0,1],$$

$$y''(t) = \frac{a_1}{t} y'(t) + \frac{a_0}{t^2} y(t) + f(t), \;\; t \in (0,1], \;\; f \in C[0,1],$$

where $\lambda, a_1, a_0 \in \mathbb{R}$.

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- Difficulty: Right hand side is not continuous on $[0,1]$. It depends on λ and a_1, a_0 whether bounded nontrivial solutions exist.

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- Difficulty: Right hand side is not continuous on $[0,1]$. It depends on λ and a_1, a_0 whether bounded nontrivial solutions exist.
- We now construct bounded solutions

$$z \in C[0,1]: \quad z(t) = z_h(t) + z_p(t).$$

Bounded solutions - 1. order scalar case

$$z'(t) = \frac{\lambda}{t} z(t), \quad t \in (0,1], \quad \lambda \in \mathbb{R}.$$

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The general solution of the homogeneous, linear problem

$$z_h'(t) = a(t) z_h(t), \quad t \in (0,1]$$

can be written as

$$z_h(t) = e^{\int_1^t a(s)ds} c,$$

where $c \in \mathbb{R}$ is an arbitrary constant.

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Thus, with $a(s) = \frac{\lambda}{s}$,

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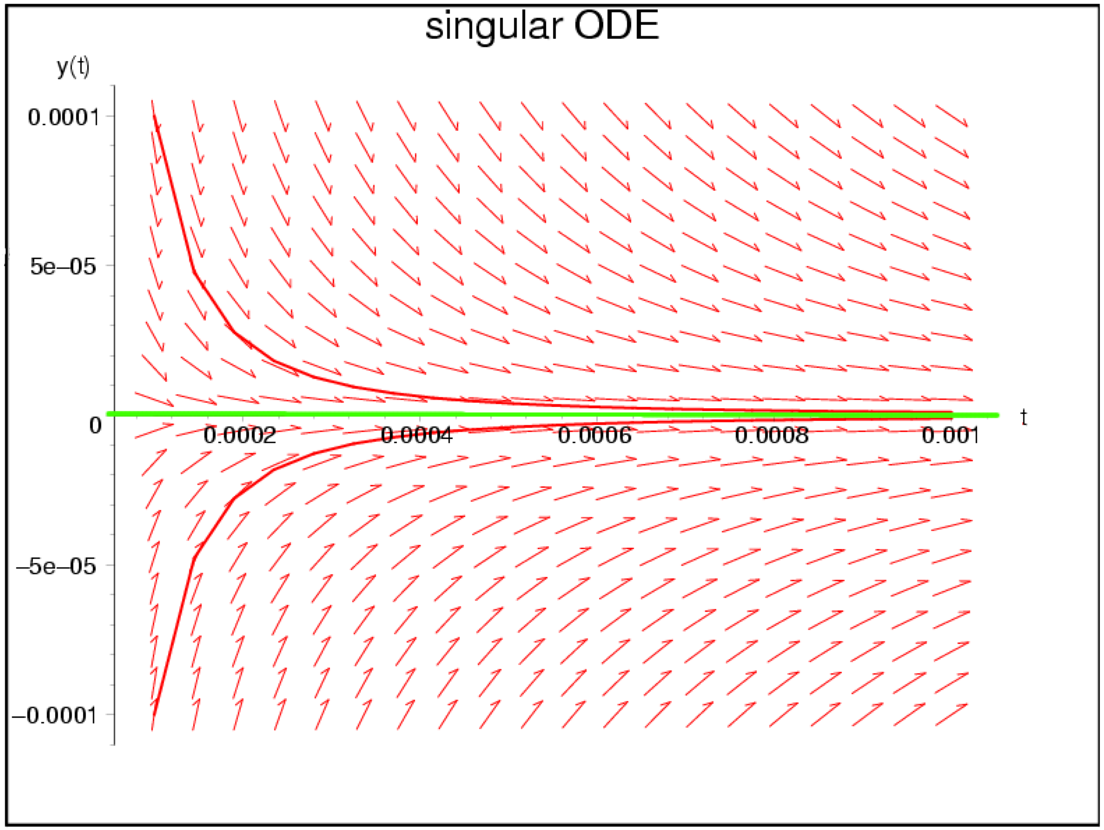
Now, we distinguish between three cases concerning λ.

Bounded solutions: $\lambda < 0$

General solution of the homogeneous problem: $z_h(t) = c\,t^{\lambda}$,

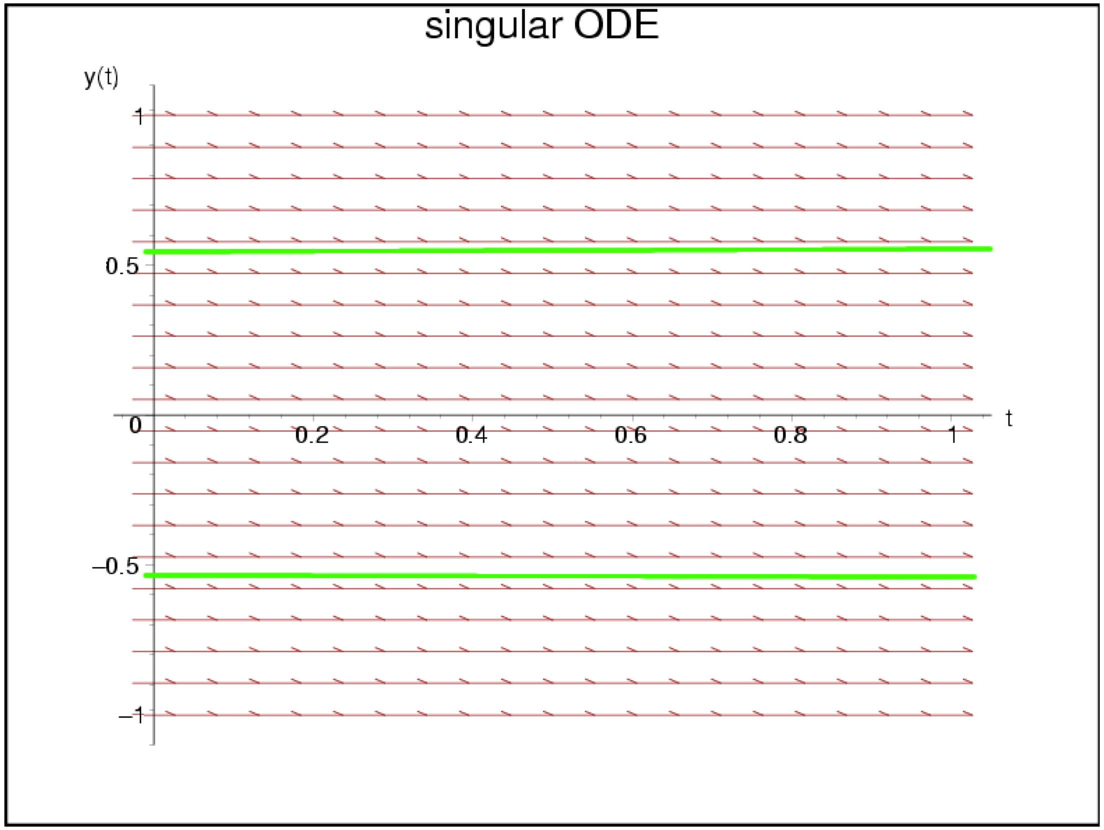
the only continuous solution is $z_h(t) = 0 \in C[0,1]$.



Bounded solutions: $\lambda = 0$

General solution of the homogeneous problem:

$$z_h(t) = c\,t^\lambda \;\Rightarrow\; z_h(t) = c \in C[0,1].$$



Bounded solutions: $\lambda > 0$

General solution of the homogeneous problem:

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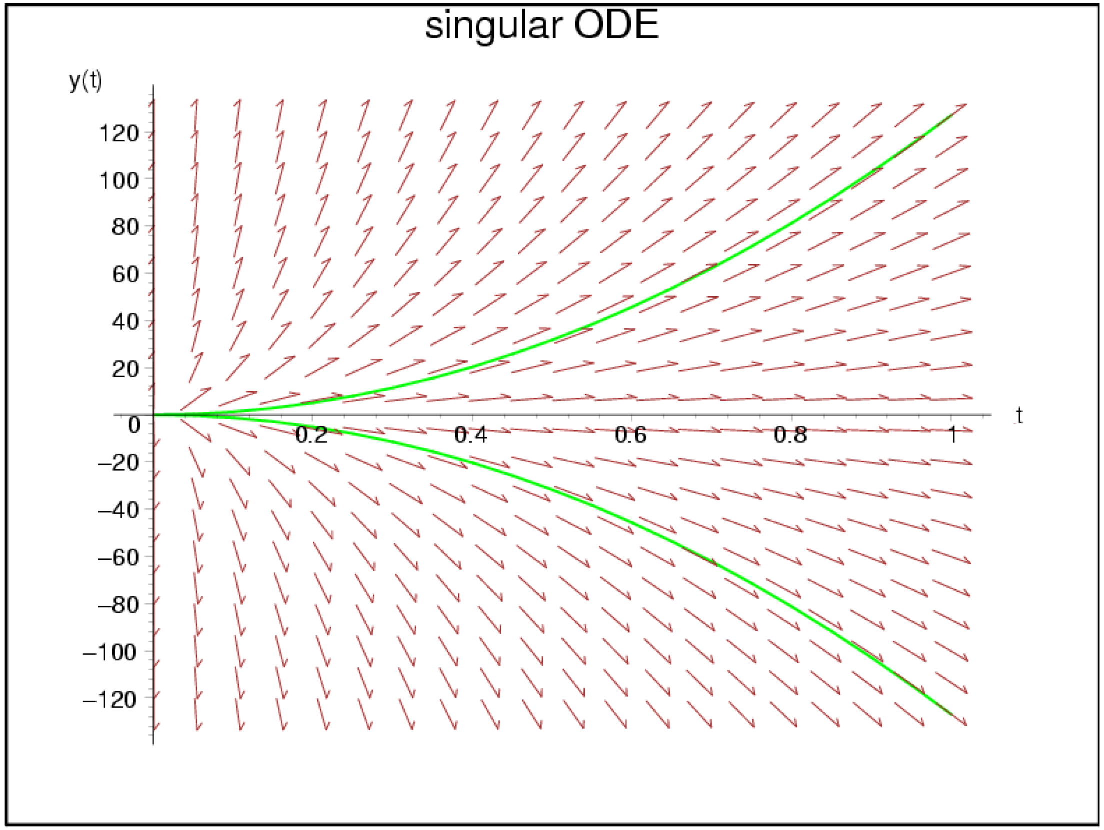

Bounded solutions - particular solution

Particular solution of the inhomogeneous problem:

$$z'(t) = \frac{\lambda}{t} z(t) + f(t), \;\; t \in (0,1], \;\; \lambda \in \mathbb{R}, \; f \in C[0,1].$$

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Variation of the constant (T1.1):

$$z_p(t) = t^{\lambda} \int_1^t s^{-\lambda} f(s) ds, \quad z_p(1) = 0.$$

Bounded solutions - particular solution

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Variation of the constant (T1.1):

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The general solution of the problem is now given by

$$z(t) = z_h(t) + z_p(t) = t^\lambda c + t^\lambda \int_1^t s^{-\lambda} f(s) ds.$$

Bounded solutions: $\lambda < 0$

Case 1: $\lambda < 0$

We first rewrite z_p (T1.2),

$$z_p(t) = t^\lambda \int_1^t s^{-\lambda} f(s) ds = t^\lambda \gamma + t^\lambda \int_0^t s^{-\lambda} f(s) ds,$$

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and transform (T1.3),

$$t^\lambda \int_0^t s^{-\lambda} f(s) ds = t \int_0^1 s^{-\lambda} f(ts) ds.$$

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$$z_p(t) = t^\lambda \int_1^t s^{-\lambda} f(s)ds = t^\lambda \gamma + t^\lambda \int_0^t s^{-\lambda} f(s)ds,$$

and transform (T1.3),

$$t^\lambda \int_0^t s^{-\lambda} f(s)ds = t \int_0^1 s^{-\lambda} f(ts)ds.$$

Then the solution of the ODE is $z(t) = t^\lambda c + t \int_0^1 s^{-\lambda} f(ts)ds,$

and $c = 0 \Leftrightarrow z(0) = 0$ means

$$z \in C[0,1] \Leftrightarrow z(t) = z_p(t) = t \int_0^1 s^{-\lambda} f(ts)ds.$$

Smoothness of solutions: $\lambda < 0$

Smoothness properties of $z(t)$: Substitute z into the DE

$$z(t) = t \int_0^1 s^{-\lambda} f(ts)ds, \qquad z'(t) = \frac{\lambda}{t} z(t) + f(t).$$

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Then

$$z'(t) = \lambda \int_0^1 s^{-\lambda} f(ts)ds + f(t), \; z^{(k+1)}(t) = \lambda \int_0^1 s^{-\lambda+k} f^{(k)}(ts)ds + f^{(k)}(t).$$

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Thus, smoothness of z depends only on smoothness of f:

If $f \in C^p[0,1]$ then $z \in C^{p+1}[0,1]$.

Summarize the case $\lambda < 0$

Theorem: Consider the following IVP:

$$z'(t) = \frac{\lambda}{t} z(t) + f(t), \quad t \in (0, 1], \quad z(0) = 0.$$

Summarize the case $\lambda < 0$

Theorem: Consider the following IVP:

$$z'(t) = \frac{\lambda}{t} z(t) + f(t), \quad t \in (0, 1], \quad z(0) = 0.$$

Then, for any $\lambda < 0$ and any $f \in C^p[0, 1]$ there exists a unique solution $z \in C^{p+1}[0, 1]$.

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This solution satisfies

$$z(t) = t \int_0^1 s^{-\lambda} f(ts) ds, \quad z'(t) = \lambda \int_0^1 s^{-\lambda} f(ts) ds + f(t),$$

and (T1.4) $$|z(t)| \le \text{const}\, t\, \|f\|, \quad |z'(t)| \le \text{const}\, \|f\|.$$

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General systems – BVPs

Vector case – variable coefficient matrix

Existence and uniqueness of continuous solutions

General systems – BVPs

Bounded solutions: $\lambda = 0$

Case 2: $\lambda = 0$

For $\lambda = 0$, the DE $z'(t) = \frac{\lambda}{t} z(t) + f(t)$ is simply $z'(t) = f(t)$.

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Its general solution reads

$$z(t) = t^\lambda d + t^\lambda \int_1^t s^{-\lambda} f(s) ds = t^\lambda c + t \int_0^1 s^{-\lambda} f(ts) ds,$$

or equivalently

$$z(t) = c + t \int_0^1 f(ts) ds, \quad z'(t) = f(t), \quad z^{(k+1)}(t) = f^{(k)}(t).$$

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$$z(t) = c + t \int_0^1 f(ts) ds, \quad z'(t) = f(t), \quad z^{(k+1)}(t) = f^{(k)}(t).$$

Clearly, $f \in C^p[0,1]$ implies $z \in C^{p+1}[0,1]$.

Summarize the case $\lambda = 0$

Theorem: Consider the following IVP:

$$z'(t) = f(t), \quad z(0) = \beta.$$

Then, for any $f \in C^p[0,1]$ there exist unique solutions $z \in C^{p+1}[0,1]$ of the IVP. This solution satisfies

$$z(t) = \beta + t \int_0^1 f(ts)ds$$

and

$$|z(t)| \leq |\beta| + t||f||, \quad |z'(t)| \leq ||f||.$$

Bounded solutions: $\lambda > 0$

Case 3: $\lambda > 0$

The general solution again reads:

$$z(t) = z_h(t) + z_p(t) = t^\lambda c + t^\lambda \int_1^t s^{-\lambda} f(s) ds.$$

Since (T1.5-T1.6),

$$|z_p(t)| \le ||f|| \begin{cases} \frac{1}{1-\lambda} t^\lambda, & \lambda < 1, \\ t|\ln t|, & \lambda = 1, \\ \frac{1}{\lambda - 1} t, & \lambda > 1. \end{cases}$$

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$z(t) \in C[0,1]$ for any $f(t) \in C[0,1]$ and $z(0) = 0$ holds.

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$z(t) \in C[0,1]$ for any $f(t) \in C[0,1]$ and $z(0) = 0$ holds.

Also for $\lambda > 1$, $z \in C^1[0,1]$ since

$$z'(t) = \lambda t^{\lambda-1} c + \lambda t^{\lambda-1} \int_1^t s^{-\lambda} f(s) ds + f(t).$$

Summarize the case $\lambda > 0$

Theorem: Consider the following TVP:

$$z'(t) = \frac{\lambda}{t} z(t) + f(t), \quad t \in (0,1], \quad z(1) = \beta.$$

Then, for any $f \in C^p[0,1]$ there exist a unique solution $z \in C[0,1] \cap C^{p+1}(0,1]$ of the TVP. This solution satisfies

$$z(t) = t^\lambda \beta + t^\lambda \int_1^t s^{-\lambda} f(s) ds,$$

and

$$|z(t)| \leq \text{const} \begin{cases} t^\lambda(|\beta| + ||f||), & \lambda < 1, \\ t(|\beta| + |\ln t|||f||), & \lambda = 1, \\ t(|\beta| + ||f||), & \lambda > 1, \end{cases} \quad |z'(t)| \leq \text{const} \begin{cases} t^{\lambda-1}(|\beta| + ||f||), & \lambda < 1, \\ (|\beta| + |\ln t|||f||), & \lambda = 1, \\ (|\beta| + ||f||), & \lambda > 1. \end{cases}$$

Moreover, for $\lambda > p+1$, $z \in C^{p+1}[0,1]$.