Workshop
Applications of Mathematics

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Infinite-horizon discrete-time optimal control

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FIRST PART : THE SETTING

Problems of Optimal Control, Infinite Horizon, Discrete Time

- Two types of Dynamical Systems:

  $x_{t+1} = f_t(x_t, u_t)$ or $x_{t+1} \leq f_t(x_t, u_t)$

  $t \in \mathbf{N}$, $x_t \in \mathbf{R}^n$, $u_t \in U_t \subset \mathbf{R}^m$, x_0 given.

- Several kinds of criterium:

1/ Maximize $\sum_{t=0}^{\infty} f_t^0(x_t, u_t)$

2/ To find $((\overline{x}_t)_t, (\overline{u}_t)_t)$ admissible s.t.

$$\liminf_{h \to \infty} \sum_{t=0}^{h} (f_t^0(\overline{x}_t, \overline{u}_t) - f_t^0(x_t, u_t)) \geq 0$$

3/ To find $((\overline{x}_t)_t, (\overline{u}_t)_t)$ admissible s.t.

$$\limsup_{h \to \infty} \sum_{t=0}^{h} (f_t^0(\overline{x}_t, \overline{u}_t) - f_t^0(x_t, u_t)) \geq 0$$

A notation : the Pontryagin Hamiltonian:

$$H_t(x, u, \lambda^0, p) := \lambda^0 f_t^0(x, u) + \langle p, f_t(x, u) \rangle$$

The aim: to obtain Pontryagin principles for these problems, i.e. results in the following form.
When $((\overline{x}_t)_t, (\overline{u}_t)_t)$ is an optimal process, then there exist $\lambda^0 \in \mathbf{R}$ and a sequence $(p_t)_t$,
$p_t \in \mathbf{R}^{n*}$, s.t.

- $(\lambda^0, (p_t)_t)$ is non zero

- $\lambda^0 \geq 0$

- $p_t = D_{x_t}(\overline{x}_t, \overline{u}_t, \lambda^0, p_{t+1})$ or
  $p_t \in \partial_{x_t} H_t(\overline{x}_t, \overline{u}_t, \lambda^0, p_{t+1})$
  (adjoint equation or adjoint inclusion)

- (a strong maximum principle)
  $\forall t \geq 0$, $\forall u_t \in U_t$,
  $H_t(\overline{x}_t, \overline{u}_t, \lambda^0, p_{t+1}) \geq H_t(\overline{x}_t, u_t, \lambda^0, p_{t+1})$

- (a weak maximum principle)
  $\forall t \geq 0$
  $\partial_{u_t} H_t(\overline{x}_t, \overline{u}_t, \lambda^0, p_{t+1}) \cap N_{U_t}(\overline{u}_t) \neq \emptyset$
  or $D_{u_t} H_t(\overline{x}_t, \overline{u}_t, \lambda^0, p_{t+1}) = 0$

A method: reduction to finite horizon

When $((\overline{x}_t)_t, (\overline{u}_t)_t)$ is an optimal process then, $\forall h \in \mathbf{N}_*$, $((\overline{x}_t)_{t \leq h}, (\overline{u}_t)_{t \leq h})$
is an optimal solution to:

$$\begin{cases} \text{Maximize} & \sum_{t=0}^{h} f_t^0(x_t, u_t) \\ \text{when} & x_{t+1} = f_t(x_t, u_t), x_0 = \overline{x}_0, x_h = \overline{x}_h \\ \text{or when} & x_{t+1} \leq f_t(x_t, u_t), x_0 = \overline{x}_0, x_h = \overline{x}_h \end{cases}$$

A first-order necessary condition of optimality gives:
$\forall h \in \mathbf{N}_*$, $\exists \lambda^{0,h} \in \mathbf{R}$, $\exists (p_t^h)_{t=1,...,h} \in (\mathbf{R}^{n*})^h$
s.t.

- $(\lambda^{0,h}, p_1^h, ..., p_h^h)$ is non zero

- $\lambda^{0,h} \geq 0$ (and $p_t^h \geq 0$ when inequation)

- $\forall t \leq h$, $p_t^h \in \partial_{x_t} H_t(\overline{x}_t, \overline{u}_t, \lambda^{0,h}, p_{t+1}^h)$

- or $\forall t \leq h-1$, $\forall u_t \in U_t$,
$H_t(\overline{x}_t, \overline{u}_t, \lambda^{0,h}, p_{t+1}^h) \geq H_t(\overline{x}_t, u_t, \lambda^{0,h}, p_{t+1}^h)$

the second step of the method:

when $h \to \infty$

$\lambda^0 = \lim_{h\to\infty} \lambda^{0,h}$, $p_t = \lim_{h\to\infty} p_t^h$

or $\lambda^0 = \lim_{h\to\infty} \lambda^{0,\alpha(h)}$, $p_t = \lim_{h\to\infty} p_t^{\alpha(h)}$
where $\alpha : \mathbf{N} \longrightarrow \mathbf{N}$ strictly increasing
(a subsequence).

- to normalize $(\lambda^{0,h}, p_1^h, ..., p_h^h)$,
- to use a Tychonov theorem (or a cantor diagonal process)
- to obtain a convergent subsequence
- the adjoint equation and the maximum principle are conserved
- the difficulty: <u>the limit is non zero</u>

Several answers to this question, for example:

- Under ($\forall t, D_{x_t} f_t(\overline{x}_t, \overline{u}_t)$ invertible), JB & H. Chebbi, JMAA 2000.

- Under ($\forall t, j, i$, $\frac{\partial f_t^j(\overline{x}_t, \overline{u}_t)}{\partial x_i^i} \geq 0$, $\frac{\partial f_t^i(\overline{x}_t, \overline{u}_t)}{\partial x_t^i} > 0$), JB in J. Nonlinear Convex Anal. 2009.

For strong maximum principles: From Boltyanski we know that it is necessary to add assumptions to obtain strong principles; it is a great difference with the continuous time.

There exist two conditions to do that:

- A condition due to P. Michel related to the "mixed problems"

- A condition due to Pchnenichnyi, Ioffe and Tihomirov: For all $t \in \mathbb{N}$, for all $x \in \Omega$, for all $u', u'' \in \mathcal{U}$, for all $\alpha \in [0, 1]$, there exists $u \in \mathcal{U}$ such that

$$\Phi(x, u) \geq (1 - \alpha)\Phi(x, u') + \alpha\Phi(x, u'')$$

$$f_t(x, u) = (1 - \alpha)f_t(x, u') + \alpha f_t(x, u'').$$

SECOND PART: THE BOUNDED PROCESSES.

$(x_t)_t$ and $(u_t)_t$ are bounded sequences. It is possible to use Nonlinear Functional Analysis in Banach spaces.

Ω is an open convex nonempty subset of $\mathbf{R}^n$, $\mathcal{U}$ is a nonempty compact subset of $\mathbf{R}^m$. $\Phi : \Omega \times \mathcal{U} \to \mathbf{R}$, $f_t : \Omega \times \mathcal{U} \to \mathbf{R}^n$, and $\beta \in (0,1)$.

$$\begin{cases} \text{Maximize } J(\underline{x}, \underline{u}) = \sum_{t=0}^{\infty} \beta^t \Phi(x_t, u_t) \\ \text{s.t. } x_{t+1} = f_t(x_t, u_t), t \in \mathbf{N} \\ x_0 = \eta \\ (\underline{x}, \underline{u}) \in \ell_\infty \text{ admissible} \end{cases}$$

A theorem of necessary conditions, JB & N. Hayek,

Let $(\underline{\widehat{x}}, \underline{\widehat{u}})$ be an optimal process. Under the assumptions

1. For all $u \in \mathcal{U}$, $\Phi(., u)$ is C^1, and for all $t \in \mathbf{N}$, $f_t(., u)$ is C^1.

2. For all $t \in \mathbf{N}$, for all $x \in \Omega$, for all $u', u'' \in \mathcal{U}$, for all $\alpha \in [0, 1]$, there exists $u \in \mathcal{U}$ such that

$$\Phi(x, u) \geq (1 - \alpha)\Phi(x, u') + \alpha\Phi(x, u'')$$

$$f_t(x, u) = (1 - \alpha)f_t(x, u') + \alpha f_t(x, u'').$$

3. For all compact set $C \subset \Omega$, there exists a constant K_C such that, for all $t \in \mathbf{N}$, for all $x, x' \in C$, for all $u \in \mathcal{U}$, $\|f_t(x, u)\| \leq K_C$ and $\|D_x f_t(x, u) - D_x f_t(x', u)\| \leq K_C.\|x - x'\|$.

4. There exists $r > 0$ such that $B(\underline{\widehat{x}}, r)$ included in the admissible states, and for all $(x_t, u_t) \in B(\widehat{x}_t, r) \times \mathcal{U}$, $\sup_{t \geq 0} |D_{x_t} f_t(x_t, u_t)| < 1$.

there exists $(p_t)_{t \geq 1} \in \ell_1(\mathbf{N}, \mathbf{R}^n)$ such that

(a) $p_t = p_{t+1} \circ D_{x_t} f_t(\widehat{x}_t, \widehat{u}_t) + \beta^t D_{x_t} \Phi(\widehat{x}_t, \widehat{u}_t)$ for all t.

(b) $\beta^t \Phi(\widehat{x}_t, \widehat{u}_t) + \langle p_{t+1}, f_t(\widehat{x}_t, \widehat{u}_t) \rangle \geq \beta^t \Phi(\widehat{x}_t, u_t) + \langle p_{t+1}, f_t(\widehat{x}_t, u_t) \rangle$ for all t, u_t.

(c) $\lim_{t \to \infty} p_t = 0$.

A theorem of sufficient conditions, JB N. Hayek:

Let $(\underline{\widehat{x}}, \underline{\widehat{u}})$ be an admissible process and $(p_t)_{t \geq 1} \in \ell_1(\mathbf{N}, \mathbf{R}^n)$. Under the assumptions

1. $\mathcal{U}$ is convex.

2. For all $u \in \mathcal{U}$, $\Phi(., u)$ is C^1, and for all $t \in \mathbf{N}$, $f_t(., u)$ is C^1.

3. $p_t = p_{t+1} \circ D_{x_t} f_t(\widehat{x}_t, \widehat{u}_t) + \beta^t D_{x_t} \Phi(\widehat{x}_t, \widehat{u}_t)$ for all t.

4. $\beta^t \Phi(\widehat{x}_t, \widehat{u}_t) + \langle p_{t+1}, f_t(\widehat{x}_t, \widehat{u}_t) \rangle \geq \beta^t \Phi(\widehat{x}_t, u_t) + \langle p_{t+1}, f_t(\widehat{x}_t, u_t) \rangle$ for all t, u_t.

5. The mappings H_t are concave with respect to (x_t, u_t) for all t.

Then $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is optimal.

Another result for bounded processes (JB, N. Hayek, F. Pekergin, N. Pekergin, Optimization 2013) for the problem $(\mathcal{P}I)$

$$\begin{cases} \text{Maximize } J(\underline{x},\underline{u}) = \sum_{t=0}^{\infty} \beta^t \Phi(x_t,u_t) \\ \text{s.t. } x_{t+1} \leq f_t(x_t,u_t), t \in \mathbf{N} \\ x_0 = \eta \\ (\underline{x},\underline{u}) \in \ell^\infty \text{ admissible} \end{cases}$$

We consider the following conditions, $(\hat{x}_t)_t, (\hat{u}_t)$ being an admissible process:

(H1) ϕ and f are C^1.

(H2) $\forall t$, $D_u f(\hat{x}_t,\hat{u}_t)$ is positive, and there exists $\gamma_1 > 0$ s.t.

$$D_{u_t} f(\hat{x}_t,\hat{u}_t).(1,1,...,1) \geq (\gamma_1,\gamma_1,...,\gamma_1).$$

(H3) $\forall t$, $D_u f(\widehat{x}_t, \widehat{u}_t)$ is negative, and there exists $\gamma_2 > 0$ s.t.

$$D_{u_t} f(\widehat{x}_t, \widehat{u}_t).(-1, -1, ..., -1) \geq (\gamma_2, \gamma_2, ..., \gamma_2).$$

(H4) When $n = d$, $\forall t$, $D_u f(\widehat{x}_t, \widehat{u}_t)$ is invertible and

$$\sup \|D_{u_t} f(\widehat{x}_t, \widehat{u}_t)^{-1}\| < +\infty.$$

(H5) There exists $\zeta > 0$ s.t. $\forall t$,

$$D_{x_t} f(\widehat{x}_t, \widehat{u}_t).(1, 1, ..., 1) \leq (\zeta, \zeta, ..., \zeta).$$

THEOREM. Let $\beta \in (0,1)$ and $\eta \in \mathbf{R}^n_{++}$. Let $(\widehat{x}_t)_t, (\widehat{u}_t)$ be a solution of $(\mathcal{P}I)$. Under (H1), we assume that (H2) or (H3) or (H4) or (H5) is fulfilled. Then there exists $(p_t)_{t\geq 1} \in \ell^1(\mathbf{N}_*, \mathbf{R}^{n*}_+$ s.t.

(AE) $\forall t \geq 1$, $p_t = p_{t+1} \circ D_{x_t} f(\widehat{x}_t, \widehat{u}_t) + \beta^t D_{x_t} \phi(\widehat{x}_t, \widehat{u}_t)$

(WMP) $\forall t \geq 0$,
$p_{t+1} \circ D_{u_t} f(\widehat{x}_t, \widehat{u}_t) + \beta^t D_{x_t} \phi(\widehat{x}_t, \widehat{u}_t) = 0$

(CCS) $\forall t \geq 0$, $\langle p_{t+1}, f(\widehat{x}_t, \widehat{u}_t) - \widehat{x}_{t+1} \rangle = 0$

(TCI) $\lim_{t \to +\infty} p_t = 0$.

Elements of the proof.

$\mathcal{P}I$ is a problem with an infinity of inequality constraints, and
$G((\widehat{x}_t)_t,(\widehat{u}_t)) := (f(x_t,u_t) - x_{t+1})_{t\geq 0}$
$G((\widehat{x}_t)_t,(\widehat{u}_t)) \in \ell^\infty(\mathbf{N},\mathbf{R}^n)_+$

$\mathcal{L}((\widehat{x}_t)_t,(\widehat{u}_t),\Lambda) := J(\widehat{x}_t)_t,(\widehat{u}_t)) + \langle \Lambda, G((\widehat{x}_t)_t,(\widehat{u}_t))\rangle$

The Lagrangian:
$\Lambda \in \ell^\infty(\mathbf{N},\mathbf{R}^n)^*$, $\Lambda = (p_{t+1})_t \oplus \Sigma$.

We use a KKT theorem

NB: $\text{int}\ell^\infty(\mathbf{N},\mathbf{R}^n)_+ \neq \emptyset$, $\text{int}\ell^p(\mathbf{N},\mathbf{R}^n)_+ = \emptyset$ if $p < +\infty$.

A essential tool: KKT theorems in ordered Banach spaces.

ABOUT TURNPIKE: a result of JB & B. Crettez, in Decision in Economics and Finance, 2007. We consider the following variational problems

$$\mathcal{P}(\eta,\beta)\begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t U(k_t, k_{t+1}) \\ \text{subject to} & U(k_t, k_{t+1}) \in \Omega, k_0 = \eta. \end{cases}$$

Theorem. We assume Ω is convex and satisfies an additional condition of "'symmetry"', and U is concave. Let $(k_t^0)_t$ be an optimal solution of $\mathcal{P}(\eta^0, \beta^0)$. which satisfies the following condition

$\forall t, D^2_{22}U(k_t^0, k_{t+1}^0) + \beta^0 . D^2_{22}U(k_{t+1}^0, k_{t+2}^0)$ invertible.

Then there exist neighborhoods of η^0 and of β^0, there exists a C^1-function K_t defined on these neighborhoods such that $(K_t(\eta,\beta))_t$ is an optimal solution of $\mathcal{P}(\eta,\beta)$ and such that $\lim_{t\to+\infty}(K_t(\eta,\beta) - k_t^0) = 0$.

ABOUT OSCILLATIONS:

We present from JB & A. Bouadi in Proceedings-Springer (to appear in 2013)

In a variational continuous-time setting, we consider the maximization of the functional

$$J(x(.)) := \int_0^{+\infty} e^{-r.t} L(t, x(t), x'(t)) dt$$

on $BC^1(\mathbf{R}_+, \mathbf{R}^n)$ the Banach space of the $x(.) \in C^1(\mathbf{R}_+, \mathbf{R}^n)$ such that $x(.)$ and $x'(.)$ are bounded on $\mathbf{R}_+$. The critical point equation $DJ(x(.)) = 0$ is equivalent to the Euler-Lagrange equation :

$$D_x L(t, x, x') = -r.D_{x'} L(t, x, x') + \frac{d}{dt} D_{x'} L(t, x, x').$$

For instance, when $L(t, x, x') = K(x, x') - \langle x, b(t) \rangle$, we obtain a forced equation

$$D_x K(x, x') + r.D_{x'} K(x, x') - \frac{d}{dt} D_{x'} K(x, x') = b(t).$$

We consider the space of the almost periodic functions which is build in the following way:

when $T > 0$, $P_T(\mathbf{R}_+, \mathbf{R}^n)$ is the space of the T-periodic functions, we take the union of all this spaces $P\mathbf{R}_+, \mathbf{R}^n) := \cup_{T>0} P_T(\mathbf{R}_+, \mathbf{R}^n)$ which included into $BC^0\mathbf{R}_+, \mathbf{R}^n)$, we consider the vector space generated by $P\mathbf{R}_+, \mathbf{R}^n)$, span$P\mathbf{R}_+, \mathbf{R}^n)$, and we take the closure for the norm of the sup, and so we obtain

$$AP^0(\mathbf{R}_+, \mathbf{R}^n) := \text{cl}(\text{span} P\mathbf{R}_+, \mathbf{R}^n).$$

and $AP^1(\mathbf{R}_+, \mathbf{R}^n)$ is the space of the $x(.) \in C^1C^1(\mathbf{R}_+, \mathbf{R}^n)$ such that $x(.)$ and $x'(.)$ belong to $AP^0(\mathbf{R}_+, \mathbf{R}^n)$. We can obtain reults of density: under assumptions, the set of the $b(.) \in AP^0$ such that there exists $x() \in AP^1$ which is a solution of the equation $D_xK(x,x') + r.D_{x'}K(x,x') - \frac{d}{dt}D_{x'}K(x,x') = b(t)$ is dense into AP^0.

THIRD PART: INFINITE DIMENSION

The state space X is an ordered Banach space.

The system is $x_{t+1} \leq f_t(x_t, u_t)$.

Reduction to the finite dimension:
using a KKT theorem in ordered Banach spaces we obtain
$\lambda_0^h \geq 0$, $p_1^h \geq 0$, ..., $p_h^h \geq 0$.

K := the positive cone of X, K^* := the positive cone of X^*

Additional assumption : there exists a uniformly positive linear functional on K^*; i.e.
$\exists \Lambda \in (\mathbf{R} \times X)^{**}, \exists a \in (0, \infty),$ s;t.
$\forall q \in \mathbf{R}_+ \times K^*, \langle \Lambda, q \rangle \geq a.\|q\|$.
(cf. M.G. Krasnoselski "' Positive solutions of operator equations"', Nordhoff, 1964)

The normalization becomes : $\langle \Lambda, (\lambda_0^h, p_1^h) \rangle = c$
then $|\lambda_0^h| + \|p_1^h\| \leq \frac{c}{a}$
and we can use the Banach-Alaoglu theorem for the *-weak topology.