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INVESTMENTS IN EDUCATION DEVELOPMENT

Streamlining the Mathematics Studies at the Faculty of Science of Palacky University in Olomouc

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The discrete case

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Definitions

- trigonometric polynomial:
  $\sum_{\lambda\in J} a_\lambda e_\lambda, \quad e_\lambda(t) = e^{i\lambda t}, \quad J \subset [0; 2\pi)$ finite
- $\underline{x} = (x_t)_{t\in\mathbb{Z}}$ is Bohr-a.p. iff there exists a sequence of t.p. $(P_n)_n$ s.t.:
  $$\|\underline{x} - P_n\|_\infty \to 0$$
- $\underline{x}$ is a.p. iff for all $\forall\varepsilon > 0$, $\exists N > 0$, $\forall k \in \mathbb{Z}$, $\exists p \in \{k, \dots, k+N\}$, $\sup_t |x_{t+p} - x_t| \leq \varepsilon$
- $\underline{x}$ is a.p. iif for each sequence $(h_n)_n$, there exists a uniformly convergent subsequence for $(x_{.+h_n})_n$.
- $\underline{x}$ is a.p. there exists $\varphi \in C^0(b\mathbb{Z})$ (necessarely unique, written $\varphi^{\underline{x}}$) s.t. $\underline{x} = \varphi \circ in$.

Link with a.p. functions

$\underline{x}$ is a.p. iff

- there exists $f \in AP^0(\mathbb{R}, E)$ such that for any $t \in \mathbb{Z}$, $f(t) = x_t$.
- The fonction $f_{\underline{x}} : \mathbb{R} \to E$ defined by $f_{\underline{x}}(t+u) = x_t + u(x_{t+1} - x_t)$ for $t \in \mathbb{Z}$, $u \in [0;1]$ is a.p.

- Mean operator:

$$\mathcal{M}\{\underline{x}\} := \lim_{T\to+\infty} \frac{1}{2T+1} \sum_{t=-T}^{T} x_t = \mathcal{M}\{f_{\underline{x}}\} = \int_{b\mathbb{Z}} \varphi^{\underline{x}} d\mu_{b\mathbb{Z}}.$$

- Set $\alpha \in [0;2\pi)$ and $t \in \mathbb{Z}$, and $\underline{e_\alpha} := (e_\alpha(t))_{t\in\mathbb{Z}}$ and finally

$$c_\alpha(\underline{x}) := \mathcal{M}\{e_{-\alpha}(t)x_t\}_t.$$

We have:

$$\underline{x} \sim \sum_{\alpha\in[0;2\pi)} c_\alpha(\underline{x})\underline{e_\alpha}$$

and Parseval's relation:

$$\mathcal{M}\{|x_t|^2\}_t = \sum_{\alpha\in[0;2\pi)} |c_\alpha(\underline{x})|^2.$$

- $AP(\mathbb{Z}, \mathbb{R}^N)$ can be endowed with the following scalar product:

$$< \underline{x}, \underline{y} >:= \mathcal{M}\{\overline{x_t} \cdot y_t\}_t$$

  and its completion is denoted by $B^2(\mathbb{Z}, \mathbb{R}^N)$ (discrete Besicovitch space). The following conditions are equivalent:
    - $\underline{x} \in B^2(\mathbb{Z}, \mathbb{R}^N)$
    - $\sum_\alpha |c_\alpha(\underline{x})|^2 < +\infty$
    - $\varphi^{\underline{x}} \in L^2(b\mathbb{Z}, \mathbb{R}^N)$
- Moreover, $\|\underline{x}\| = \|\varphi^{\underline{x}}\|$. $B^2(\mathbb{Z}, \mathbb{R}^N)$ and $L^2(b\mathbb{Z}, \mathbb{R}^N)$ are similar and be viewen as spaces where Harmonic Synthesis is realized.

u.a.p. sequences w.r.t. a parameter

- Assume P is compact or $P = \cup_p K_p$ with K_p compact for all p and $K_p \subset \mathrm{Int} K_{p+1}$. Let us consider $A : \mathbb{Z} \times P \to \mathbb{R}^N$. We say that A is u.a.p. ($A \in APU(\mathbb{Z}, P, \mathbb{R}^N)$) if for any $K \subset P$ compact and any $\varepsilon > 0$ we have:

$$\exists N \in \mathbb{N},\ \forall m \in \mathbb{Z},\ \exists p \in \{m, ..., m+N\},\ \forall t \in \mathbb{Z},$$

$$\sup_{(t,\alpha)\in\mathbb{Z}\times K} |A(t+p,\alpha) - A(t,\alpha)| \leq \varepsilon.$$

- $APU(\mathbb{Z}, P, \mathbb{R}^N)$, $AP(\mathbb{Z}, C^0(P, \mathbb{R}^N))$ and $C^0(b\mathbb{Z} \times P, \mathbb{R}^N)$ are isomorphic as Fréchet spaces.
- When $A \in APU(\mathbb{Z}, \mathbb{R}^k, \mathbb{R}^N)$, $\mathcal{N}_A \in C^0(AP(\mathbb{Z}, \mathbb{R}^k), AP(\mathbb{Z}, \mathbb{R}^N))$.

Linear a.p. discrete equations

- Let $\underline{x}$ be a.p. Then the sequence $(\sum_{j=0}^{t} x_j)_t$ is a.p. iff it is bounded.
- Given $\underline{b} \in AP^0(\mathbb{R}, \mathbb{R}^N)$, any solution of:

$$x_{t+1} = Ax_t + b_t$$

  is almost periodic if and only if it is bounded.
- Moreover, if A has no eigenvalue with modulus 1, there exists a unique a.p. solution.
- As a corollary, consider:

$$\sum_{j=0}^{p} a_j x_{t+j} = b_t$$

  with $a_j \in \mathbb{R}$, $\underline{b} \in AP(\mathbb{Z}, \mathbb{R})$ s.t. there exists $s \in \{0, ..., p\}$:

$$|a_s| > \sum_{j \neq s} |a_j|.$$

  Then there exists a unique solution in $AP(\mathbb{Z}, \mathbb{R})$.

A nonlinear case

- Consider

$$A_t(x_t, ..., x_{t+p}) = 0$$

where $A : b\mathbb{Z} \times (\mathbb{R}^N)^{p+1} \to \mathbb{R}^N$ satisfy:

  - $D_2 A$ exists, A and $D_2 A$ are Caratheodory.
  - $(A_t(0))_t \in L^2(b\mathbb{Z}; \mathbb{R}^N)$ and $(DA_t(0))_t \in L^2(b\mathbb{Z}; \mathbb{R}^N \times \mathbb{R}^N)$.
  - There exists $c > 0$ such that for all t, A_t and DA_t are $c-$Lipschitzian
  - $\exists \gamma > 0$, $\exists s \in \{0, ..., p\}$, $\exists \epsilon \in \{-1; 1\}$, $\forall v \in \mathbb{R}^N$, $\forall (t, \alpha) \in \mathbb{Z} \times (\mathbb{R}^N)^{p+1}$:

$$\epsilon \ v^T D_{s+1} A_t(\alpha) v \geq \left(\gamma + \sum_{j \neq s} \sup_{(\tau, \beta)} \| D_{j+1} A_\tau(\beta) \|_{\mathcal{L}} \right) |v|^2$$

  where v^T is the transpose of v.

- Then there exists a solution to the equation.

A variational principle in AP^0

- Let us consider $L \in APU(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$ s.t. for all t, $L(t,.)$ is C^1 and $D_2 L \in APU(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$.
- We introduce

$$J : \underline{x} \mapsto \mathcal{M} \left\{ L_t(x_t, \dots, x_{t+p}) \right\}_t$$

and the equation:

$$\sum_{j=0}^{p} D_{j+1} L_{t-j}(x_{t-j}, \dots, x_{t-j+p}) = 0.$$

- Then $\underline{x}$ is a solution of the equation in $AP(\mathbb{Z}, \mathbb{R}^N)$ iff $\underline{x}$ is a critical point of J in $AP(\mathbb{Z}, \mathbb{R}^N)$.
- if L_t is concave (or convex) for all t, then the set of solutions in $AP(\mathbb{Z}, \mathbb{R}^N)$ is a closed convex subset of $AP(\mathbb{Z}, \mathbb{R}^N)$. There is at most one solution if L_t is strictly concave (or convex).

A variational principle in B^2

- We consider $L : b\mathbb{Z} \times \mathbb{R}^k \to \mathbb{R}$ a Caratheodory function with $D_2 L$ Caratheodory and:

$$|L(t,x)| \leq a|x|^2 + b(t),$$

$$|D_2 L(t,x)| \leq c|x| + d(t),$$

with $a > 0$, $c > 0$, $b, d \in L^1(b\mathbb{Z})$.

- We introduce

$$J : \underline{x} \mapsto \mathcal{M}\left\{L_t(x_t, \dots, x_{t+p})\right\}_t$$

and the equation:

$$\sum_{j=0}^{p} D_{j+1} L_{t-j}(x_{t-j}, \dots, x_{t-j+p}) = 0.$$

- Then $\underline{x}$ is a solution of the equation in $B^2(\mathbb{Z}, \mathbb{R}^N)$ iff $\underline{x}$ is a critical point of J in $B^2(\mathbb{Z}, \mathbb{R}^N)$.

A concave coercive case in $B^2(\mathbb{Z}, \mathbb{R}^N)$

- If L satisfy:
  - $L(t, x) \leq a|x|^2 + b(t), \quad a > 0, \quad b \in L^1(b\mathbb{Z})$
  - For all $t \in \mathbb{Z}$, L_t is concave.
  - There exists $(\alpha_i)_i \in \mathbb{R}^{p+1}$ with $\sum_i \alpha_i > 0$ and $\gamma \in L^1(b\mathbb{Z}; \mathbb{R})$ such that for all $t \in \mathbb{Z}$:

$$L_t(x_1, \cdots, x_p) \leq -\left(\sum_{i=1}^{p} \alpha_i \mid x_i \mid^2\right) + \gamma(t).$$

- Then there exists a $B^2(\mathbb{Z}; \mathbb{R}^N)$ a.p. solution.

A quasi-linear case

Let I be a finite subset of $\mathbb{Z}$, for any $\tau \in I$, $(a_{t,\tau})_t \in \ell^\infty(\mathbb{Z};\mathbb{R})$, $\tau_1, \cdots, \tau_p \in \mathbb{Z}$ distincts integers, $\phi \in APU(\mathbb{Z};(\mathbb{R}^N)^p;\mathbb{R}^N)$. We assume that:

- $\alpha := \sum_{\tau \in I} \inf_t (a_{t,\tau}^2) - \sum_{\tau \neq \tau'} \|a_{\cdot,\tau}\|_{\ell^\infty} \|a_{\cdot,\tau'}\|_{\ell^\infty} > 0.$
- $\forall (t, x_1, \cdots, x_p, y_1, \cdots, y_p) \in \mathbb{Z} \times (\mathbb{R}^N)^{2p}$

$$\|\phi_t(x_1, \cdots, x_p) - \phi_t(y_1, \cdots, y_p)\| \leq \lambda \sum_{j=1}^{p} \|x_j - y_j\|$$

- $p\lambda < \alpha^{1/2}$.

Then, the equation:

$$\sum_{\tau \in I} a_{t,\tau} x_{t+\tau} - \phi_t(x_{t+\tau_1}, \cdots, x_{t+\tau_p}) = 0$$

has a solution in $B^2(\mathbb{Z};\mathbb{R}^N)$.

Stepanov a.p. sequences

- $$\|\underline{x}\|_{\mathcal{S}^1_{\mathrm{T}}} := \sup_{n\in\mathbb{Z}} \left(\frac{1}{T} \sum_{k=n}^{n+T-1} |x_k|_E \right) \in [0,\infty], \quad \text{for } T \in \mathbb{N},$$

  and:

  $$\mathcal{S}^1_{\mathrm{T}} := \{\underline{x} \mid \|\underline{x}\|_{\mathcal{S}^1_{\mathrm{T}}} < \infty\}.$$

- For $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$:

  $$\|f\|_{\mathrm{S}^1_1} := \sup_{a\in\mathbb{R}} \left(\int_a^{a+1} |f(t)|_E \, dt \right) \in [0,\infty],$$

  $$\|f\|_{\mathrm{S}^1_{1,\mathbb{Z}}} := \sup_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f(t)|_E \, dt \right) \in [0,\infty],$$

  $$\mathrm{S}^1_1 := \{f \mid \|f\|_{\mathrm{S}^1_1} < \infty\} \quad \text{and} \quad \mathrm{S}^1_{1,\mathbb{Z}} := \{f \mid \|f\|_{\mathrm{S}^1_{1,\mathbb{Z}}} < \infty\}.$$

Relations

For all $\underline{x} \in E^{\mathbb{Z}}$, $T \in \mathbb{N}$ and $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$:

- $\|\underline{x}\|_{\mathcal{S}^1_1} = \|\underline{x}\|_\infty$;
- $\frac{1}{T}\|\underline{x}\|_\infty \leq \|\underline{x}\|_{\mathcal{S}^1_{\mathrm{T}}} \leq \|\underline{x}\|_\infty$;
- $\|f\|_{\mathrm{S}^1_{1,\mathbb{Z}}} \leq \|f\|_{\mathrm{S}^1_1} \leq 2\|f\|_{\mathrm{S}^1_{1,\mathbb{Z}}}$;
- $\|f\|_{\mathrm{S}^1_1} \leq \|f\|_\infty$;
- $\|f_{\underline{x}}\|_\infty = \|\underline{x}\|_\infty$.

Stepanov a.p. sequences

- The following properties are equivalent:
  - $\underline{x} \in \ell^{\infty}$;
  - $\underline{x} \in \mathcal{S}^1_{\mathrm{T}}$, for any $T \in \mathbb{N}$;
  - $\underline{x} \in \mathcal{S}^1_{\mathrm{T}}$, for some $T \in \mathbb{N}$;
  - $f_{\underline{x}} \in \mathrm{S}^1_1$;
  - $f_{\underline{x}} \in \mathrm{S}^1_{1,\mathbb{Z}}$;
  - $f_{\underline{x}} \in L^{\infty}$.
- Moreover, all the norms $\|\cdot\|_{\infty}$, $\|\cdot\|_{\mathcal{S}^1_{\mathrm{T}}}$, $\underline{x} \to \|f_{\underline{x}}\|_{\infty}$, $\underline{x} \to \|f_{\underline{x}}\|_{\mathrm{S}^1_1}$ are equivalent.
- $\underline{x}$ is Stepanov a.p. iff it is a.p. iff $f_{\underline{x}}$ is a.p. iff $f_{\underline{x}}$ is S_{ap}.