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Fractional BVPs with strong time singularities and limit properties of their solutions

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Overview

1. Motivation

In the literature solutions of the singular differential equations

$$y'' = \frac{\rho}{t} y' + f(t, y, y')$$

are considered in the class $C^1(J)$, $J = [0, T]$. Here $\rho \neq 0$ and f is a continuous or a Carathéodory function. The above equation has a singularity in the time variable at $t = 0$ because $\int_0^T (1/t)\,\mathrm{d}t = \infty$

N. Anderson, A.M. Arthurs, Bull. Math. Biol. 43 (1981) 341.

N. Tosaka, S. Miyake, J. College of Industrial Technology, Nihon University 15 (1982) 69.
$y'' = -\frac{2}{t} y' - ce^{-y}$, $y'(0) = 0$, $dy(1) + y'(1) = 0$, $c > 0$, $d > 0$

W.F. Ford, J.A. Pennline, Nonlinear Anal. 71 (2009) 1059–1072
$y'' = \frac{m}{t} y' + f(t, y)$, $y'(0) = 0$, $Ay(1) + By'(1) = C$, $A > 0$, $B, C \geq 0$

I. Rachůnková, S. Staněk, E. Weinmüller and M. Zenz, Bound. Value Probl. 2009 (2009) 28. Article ID 905769.
$\rho > 0$, $y'' = \frac{\rho}{t} y' + f(t, y, y')$ periodic problem

I. Rachůnková, S. Staněk, E. Weinmüller and M. Zenz,Comput. Math. Appl. 60 (2010) 722–733.
$\rho < 0$, $y'' = \frac{\rho}{t} y' + f(t, y, y')$ Neumann problem

A. Feichtinger, I. Rachůnková, S. Staněk and E. Weinmüller, Comput. Math. Appl. 62 (2011) 2058–2070.
$\rho > 0$, $y'' = \frac{\rho}{t} y' + f(t, y, y')$ periodic problem

"More time singular" is the equation

$$y'' = \frac{\rho}{t^\gamma} y' + \frac{f(t,y)}{t^\mu},$$

where $\rho \neq 0$, $\gamma \in [1,\infty)$ and $\mu \in [0,\gamma]$. Such equation is the special case of

$$u''(t) = \rho p(t) u'(t) + p(t) a(t) f(t, u(t)), \tag{1}$$

where $f \in C(J \times \mathbb{R})$ and p, a satisfy

(H_1) $p \in C(0,T]$, $a \in C(J)$, $p > 0$, $a > 0$ on $(0,T]$ and $\int_0^T p(t)\,\mathrm{d}t = \infty$.

The fractional analog of (1) is the fractional differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} {}^c\!D^\alpha u(t) = \rho p(t) {}^c\!D^\alpha u(t) + p(t) a(t) f(t, u(t)),$$

where $\alpha \in (0,1)$ and ${}^c\!D$ denotes the Caputo fractional derivative.

Let $\rho \neq 0$ and $\{\alpha_n\} \subset (0,1)$ be convergent and $\lim_{n\to\infty} \alpha_n = 1$. We are interesting in

(i) the existence and uniqueness of solutions to fractional BVP

$$\frac{\mathrm{d}}{\mathrm{d}t} {}^cD^{\alpha_n} u(t) = \rho p(t) {}^cD^{\alpha_n} u(t) + p(t)a(t)f(t, u(t)), \quad n \in \mathbb{N}, \tag{2}$$

$$\textit{if } \rho < 0 \qquad \left.\begin{aligned} u(0) = {}^cD^{\alpha_n} u(t)|_{t=T} \quad \text{if } a(0) > 0, \quad n \in \mathbb{N}, \\ {}^cD^{\alpha_n} u(t)|_{t=0} = 0, \quad u(0) = {}^cD^{\alpha_n} u(t)|_{t=T} \quad \text{if } a(0) = 0, \quad n \in \mathbb{N}. \end{aligned}\right\} \tag{3}$$

$$\textit{if } \rho > 0 \qquad \left.\begin{aligned} u(0) = 0, \quad u(T) = 0 \quad \text{if } a(0) > 0, \quad n \in \mathbb{N}, \\ u(0) = 0, \quad u(T) = 0, \quad u'(0) = 0 \quad \text{if } a(0) = 0, \quad n \in \mathbb{N}. \end{aligned}\right\} \tag{4}$$

We say that a function $u : J \to \mathbb{R}$ is *a solution of* (2) if ${}^cD^{\alpha_n} u \in C(J) \cup C^1(0, T]$ and (2) is satisfied for $t \in (0, T]$.

(ii) the relations between solutions of (2), (3) ((2), (4)) and solutions to differential BVP

$$u''(t) = \rho p(t)u'(t) + p(t)a(t)f(t, u(t)), \tag{5}$$

$$\text{if } \rho < 0 \qquad \left.\begin{array}{c} u(0) = u'(T) \quad \text{if } a(0) > 0, \\ u(0) = u'(T), \quad u'(0) = 0 \quad \text{if } a(0) = 0. \end{array}\right\} \tag{6}$$

$$\text{if } \rho > 0 \qquad \left.\begin{array}{c} u(0) = 0, \quad u(T) = 0 \quad \text{if } a(0) > 0, \\ u(0) = 0,\ u(T) = 0, \quad u'(0) = 0 \quad \text{if } a(0) = 0. \end{array}\right\} \tag{7}$$

A function $u \in C^1(J) \cup C^2(0, T]$ is called *a solution of* (5) if (5) holds for $t \in (0, T]$.

2. Fractional calculus

The Riemann-Liouville fractional integral $I^\gamma x$ of order $\gamma > 0$ of a function $x : J \to \mathbb{R}$ is defined as

$$I^\gamma[x](t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} x(s)\,\mathrm{d}s,$$

where Γ is the Euler gamma function.
The Caputo fractional derivative ${}^cD^\gamma x$ of order $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $x : J \to \mathbb{R}$ is given as

$${}^cD^\gamma x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} \left(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} s^k \right) \mathrm{d}s,$$

where $n = [\gamma] + 1$ and $[\gamma]$ means the integral part of the fractional number γ.
For $\gamma \in \mathbb{N}$, ${}^cD^\gamma x(t) = x^{(\gamma)}(t)$.

In particular, if $\alpha \in (0,1)$, then

$$^{c}D^{\alpha}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}(x(s)-x(0))\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha}[x-x(0)](t).$$

and the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{c}D^{\alpha}u(t) = \rho p(t){}^{c}D^{\alpha}u(t) + p(t)a(t)f(t,u(t))$$

can be written as

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}(u(s)-u(0)),\mathrm{d}s \quad &= \quad \rho p(t)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}(u(s)-u(0)),\mathrm{d}s \\ &\quad + p(t)a(t)f(t,u(t))\end{aligned}$$

LEMMA 1.

(i) $I^\gamma : C(J) \to C(J)$ *for* $\gamma > 0$,

(ii) $I^\beta I^\gamma[x](t) = I^{\beta+\gamma}[x](t)$ *for* $t \in J$ *and* $x \in C(J)$, *where* $\beta, \gamma \in (0, \infty)$,

(iii) *if* $\gamma \in (0,1)$ *and* $x, {}^cD^\gamma x \in C(J)$, *then* $I^\gamma\, {}^cD^\gamma x(t) = x(t) - x(0)$ *for* $t \in J$,

(iv) ${}^cD^\gamma I^\gamma[x](t) = x(t)$ *for* $t \in J$, $x \in C(J)$ *and* $\gamma > 0$.

3. Problem with $\rho < 0$
3a. Auxiliary problem

Let p satisfy (H_1), that is, $p \in C(0, T]$, $p > 0$ on $(0, T]$, $\int_0^T p(t)\,\mathrm{d}t = \infty$. Let

$$v(t) = \int_t^T p(s)\,\mathrm{d}s, \quad t \in (0, T].$$

Let $r \in C(J)$. Then

$$\left| \int_0^t p(s) e^{\rho v(s)} r(s)\,\mathrm{d}s \right| \le \frac{\|r\|}{|\rho|} e^{\rho v(t)}, \quad t \in (0, T],$$

$$w(t) = \begin{cases} e^{-\rho v(t)} \displaystyle\int_0^t p(s) e^{\rho v(s)} r(s)\,\mathrm{d}s & \text{if } t \in (0, T], \\ \dfrac{r(0)}{|\rho|} & \text{if } t = 0, \end{cases}$$

is continuous on J and $\|w\| \le \|r\|/|\rho|$.

LEMMA 2.

Let (H_1) hold and let $r \in C(J)$. Then the function u,

$$u(t) = \int_0^t \frac{(t-s)^{\alpha_n - 1}}{\Gamma(\alpha_n)} \underbrace{e^{-\rho v(s)} \left(\int_0^s p(\xi) e^{\rho v(\xi)} r(\xi)\, \mathrm{d}\xi \right)}_{w(s)} \mathrm{d}s + \int_0^T p(s) e^{\rho v(s)} r(s)\, \mathrm{d}s \tag{8}$$

is the unique solution of the problem

$$\frac{\mathrm{d}}{\mathrm{d}t} {}^c D^{\alpha_n} u(t) = \rho p(t) {}^c D^{\alpha_n} u(t) + p(t) r(t), \quad n \in \mathbb{N},$$

$$\left.\begin{array}{c} u(0) = {}^c D^{\alpha_n} u(t)|_{t=T} \quad \text{if } a(0) > 0, \quad n \in \mathbb{N}, \\ {}^c D^{\alpha_n} u(t)|_{t=0} = 0, \; u(0) = {}^c D^{\alpha_n} u(t)|_{t=T} \quad \text{if } a(0) = 0, \quad n \in \mathbb{N}. \end{array}\right\}$$

Proof. Let u be a solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{c}D^{\alpha_n}u(t) = \rho p(t){}^{c}D^{\alpha_n}u(t) + p(t)r(t).$$

Then ${}^{c}D^{\alpha_n}u \in C(J) \cup C^1(0,T]$ and the function $y(t) = {}^{c}D^{\alpha_n}u(t)$, $t \in J$, is a solution of the linear first-order differential equation

$$y' = \rho p(t)y + p(t)r(t) \tag{9}$$

in the set $C(J)$. Since $\{ce^{-\rho v}; c \in \mathbb{R}\}$ is the set of all solutions to $y' = \rho p(t)y$ on $(0,T]$ and $w \in C(J)$ is a solution of (9), we see that $z(t) = ce^{-\rho v} + w(t)$, $c \in \mathbb{R}$, is the general solution of (9) on $(0,T]$. Due to $\lim_{t\to 0} e^{-\rho v(t)} = \infty$, we have

$${}^{c}D^{\alpha_n}u(t) = w(t) \quad \text{for } t \in J \tag{10}$$

If $r(0) = 0$, then $w(0) = 0$, and therefore ${}^{c}D^{\alpha_n}u(t)|_{t=0} = 0$.
We now apply I^{α_n} to (10) and get

$$u(t) = I^{\alpha_n}[w](t) + u(0) \quad \text{for } t \in J.$$

If u satisfies (3), then $u(0) = w(T)$. Hence $u(0) = \int_0^T p(s)e^{\rho v(s)}r(s)\,\mathrm{d}s$.

3b. Solvability of fractional BVPs

Let $\mathcal{H}_{\alpha_n}$ be an operator acting on $C(J)$ given by

$$(\mathcal{H}_{\alpha_n}x)(t) = \int_0^t \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)} e^{-\rho v(s)} \left(\int_0^s p(\xi) e^{\rho v(\xi)} a(\xi) f(\xi, x(\xi))\,\mathrm{d}\xi \right) \mathrm{d}s + \int_0^T p(s) e^{\rho v(s)} a(s) f(s, x(s))\,\mathrm{d}s,$$

LEMMA 3.

Let (H_1) hold and let $f \in C(J \times \mathbb{R})$. Then
(i) $\mathcal{H}_{\alpha_n} : C(J) \to C(J)$ and $\mathcal{H}_{\alpha_n}$ is completely continuous,
(ii) u_n is a solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^cD^{\alpha_n}u(t) = \rho p(t){}^cD^{\alpha_n}u(t) + p(t)a(t)f(t,u(t)), \quad n \in \mathbb{N},$$

$$\left.\begin{array}{r} u(0) = {}^cD^{\alpha_n}u(T)|_{t=T} \quad \text{if } a(0) > 0, \ \ n \in \mathbb{N}, \\ {}^cD^{\alpha_n}u(t)|_{t=0} = 0, \ \ u(0) = {}^cD^{\alpha_n}u(t)|_{t=T} \quad \text{if } a(0) = 0, \ \ n \in \mathbb{N}. \end{array}\right\}$$

if and only if it is a fixed point of $\mathcal{H}_{\alpha_n}$.

THEOREM 1 (Existence theorem).

Let (H_1) *hold and let*
(H_2) $f \in C(J \times \mathbb{R})$ *and there exists a positive constant* S *such that for* $t \in J$ *and* $|x| \le S$*, the estimate*

$$a(t)|f(t,x)| \le \left(\frac{\max\{1,T\}}{\Delta} + 1\right)^{-1} |\rho| S \tag{11}$$

is fulfilled, where $\Delta = \min\{\Gamma(\tau) : 1 \le \tau \le 2\}$ $(\doteq 0.88)$.
Then for each $n \in \mathbb{N}$ *problem*

$$\frac{\mathrm{d}}{\mathrm{d}t} {}^{c}D^{\alpha_n} u(t) = \rho p(t) {}^{c}D^{\alpha_n} u(t) + p(t)a(t)f(t,u(t)), \quad n \in \mathbb{N},$$

$$\left.\begin{array}{c} u(0) = {}^{c}D^{\alpha_n} u(t)|_{t=T} \quad \text{if } a(0) > 0, \quad n \in \mathbb{N}, \\ {}^{c}D^{\alpha_n} u(t)|_{t=0} = 0, \; u(0) = {}^{c}D^{\alpha_n} u(t)|_{t=T} \quad \text{if } a(0) = 0, \quad n \in \mathbb{N}. \end{array}\right\}$$

has at least one solution u_n *and*

$$\|u_n\| \le S \quad \text{for } n \in \mathbb{N}.$$

REMARK 1.
Let $f \in C(J \times \mathbb{R})$ and

$$a(t)|f(t,x)| \le \gamma(|x|), \quad t \in J, \; x \in \mathbb{R},$$

where $\gamma \in C[0,\infty)$, γ is nondecreasing and $\lim_{x\to\infty} \gamma(x)/x = 0$. Then f satisfies condition (H_2).

The proof of Theorem 1 is based on the Rothe fixed point theorem

LEMMA.
Let X be a normed space and $\mathcal{M}$ be a bounded convex subset of X with $\partial\mathcal{M}$ the boundary of $\mathcal{M}$. Let $\mathcal{F} : \overline{\mathcal{M}} \to X$ be a compact operator such that $\mathcal{F}$ maps $\partial\mathcal{M}$ into $\mathcal{M}$. Then $\mathcal{F}$ has a fixed point.

THEOREM 2 (Uniqueness theorem).
Let (H_1) hold. Let $f \in C(J \times \mathbb{R})$ and

$$|f(t,x) - f(t,y)| \le K|x-y| \quad \text{for } t \in J,\ x, y \in \mathbb{R},$$

where

$$K < \left(\frac{\max\{1, T\}}{\Delta} + 1\right)^{-1} \frac{|\rho|}{\|a\|}. \tag{12}$$

Then the problem

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{c}D^{\alpha_n}u(t) = \rho p(t){}^{c}D^{\alpha_n}u(t) + p(t)a(t)f(t,u(t)), \quad n \in \mathbb{N},$$

$$\left.\begin{array}{c} u(0) = {}^{c}D^{\alpha_n}u(t)|_{t=T} \quad \text{if } a(0) > 0, \quad n \in \mathbb{N}, \\ {}^{c}D^{\alpha_n}u(t)|_{t=0} = 0, \ u(0) = {}^{c}D^{\alpha_n}u(t)|_{t=T} \quad \text{if } a(0) = 0, \quad n \in \mathbb{N}. \end{array}\right\}$$

has a unique solution for each $n \in \mathbb{N}$.

EXAMPLE 1.

Let $x_0 > 0$, $\mu \geq 1$, $\gamma \in [0, \mu]$ and $r, q \in C(J)$ be such that
$\|r\| + \|q\| \leq \left(\frac{\max\{1,T\}}{\Delta} + 1\right)^{-1} T^{\gamma-\mu}|\rho| x_0$.
Then (H_2) is satisfied for $S = x_0$. Theorem 1 guarantees that for each $n \in \mathbb{N}$ the problem

$$\frac{\mathrm{d}}{\mathrm{d}t} {}^cD^{\alpha_n} u(t) = \frac{\rho}{t^\mu} {}^cD^{\alpha_n} u(t) + \frac{r(t) + q(t)e^{u(t)-x_0}}{t^\gamma}, \tag{13}$$

$$\left.\begin{aligned} u(0) = {}^cD^{\alpha_n} u(t)|_{t=T} \quad &\text{if } \gamma = \mu, \\ {}^cD^{\alpha_n} u(t)|_{t=0} = 0,\ u(0) = {}^cD^{\alpha_n} u(t)|_{t=T} \quad &\text{if } \gamma \in [0, \mu) \end{aligned}\right\} \tag{14}$$

has a solution u_n and $\|u_n\| \leq x_0$.

EXAMPLE 2.

Let $r, q \in C(J)$ and let $\gamma \in (0,1)$, $\mu \geq 1$. Combining Theorem 1 with Remark 1, there exists $S > 0$ such that for each $n \in \mathbb{N}$ the problem

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}{}^{c}D^{\alpha_n}u(t) = \frac{\rho}{t^{\mu}}{}^{c}D^{\alpha_n}u(t) + r(t) + q(t)|u(t)|^{\gamma}, \\ {}^{c}D^{\alpha_n}u(t)|_{t=0} = 0, \quad u(0) = {}^{c}D^{\alpha_n}u(t)|_{t=T} \end{aligned}\right\} \tag{15}$$

has a solution u_n and $\|u_n\| \leq S$.

3c. The relation between solutions of fractional and differential problems

THEOREM 3.

Let (H_1) and (H_2) hold. Let u_n be a solution of problem

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{c}D^{\alpha_n}u(t) = \rho p(t){}^{c}D^{\alpha_n}u(t) + p(t)a(t)f(t,u(t)),$$

$$\left.\begin{array}{c} u(0) = {}^{c}D^{\alpha_n}u(T)|_{t=T} \quad \textit{if } a(0) > 0, \\ {}^{c}D^{\alpha_n}u(t)|_{t=0} = 0, \quad u(0) = {}^{c}D^{\alpha_n}u(t)|_{t=T} \quad \textit{if } a(0) = 0, \end{array}\right\}$$

and $\|u_n\| \le S$ for $n \in \mathbb{N}$ with $S > 0$ from Theorem 1.
Then there exist a subsequence $\{u_{\ell_n}\}$ of $\{u_n\}$ and a solution u of problem

$$u''(t) = \rho p(t)u'(t) + p(t)a(t)f(t,u(t)),$$

$$\left.\begin{array}{c} u(0) = u'(T) \quad \textit{if } a(0) > 0, \\ u(0) = u'(T), \quad u'(0) = 0 \quad \textit{if } a(0) = 0. \end{array}\right\}$$

such that

$$\lim_{n\to\infty} \|u_{\ell_n} - u\| = 0, \quad \lim_{n\to\infty} \|{}^{c}D^{\alpha_{\ell_n}}u_{\ell_n} - u'\| = 0.$$

PROOF. **Step 1.** By Theorem 1, $\|u_n\| \le S$ for $n \in \mathbb{N}$. We prove that $\{u_n\}$ is equicontinuous on J. Hence there exist a subsequence $\{u_{\ell_n}\}$ of $\{u_n\}$ and $u \in C(J)$ such that

$$\lim_{n\to\infty} \|u_{\ell_n} - u\| = 0.$$

Step 2. Since $u_{\ell_n} = \mathcal{H}_{\ell_n} u_{\ell_n}$, we have

$$\begin{aligned} u_{\ell_n}(t) &= \int_0^t \frac{(t-s)^{\alpha_{\ell_n}-1}}{\Gamma(\alpha_{\ell_n})} e^{-\rho v(s)} \left(\int_0^s p(\xi) e^{\rho v(\xi)} a(\xi) f(\xi, u_{\ell_n}(\xi))\, \mathrm{d}\xi \right) \mathrm{d}s \\ &\quad + \int_0^T p(s) e^{\rho v(s)} a(s) f(s, u_{\ell_n}(s))\, \mathrm{d}s, \quad n \in \mathbb{N}. \end{aligned}$$

Letting $n \to \infty$

$$\begin{aligned} u(t) &= \int_0^t e^{-\rho v(s)} \left(\int_0^s p(\xi) e^{\rho v(\xi)} a(\xi) f(\xi, u(\xi))\, \mathrm{d}\xi \right) \mathrm{d}s \\ &\quad + \int_0^T p(s) e^{\rho v(s)} a(s) f(s, u(s))\, \mathrm{d}s, \quad t \in J, \end{aligned} \tag{16}$$

by the Lebesgue dominated convergence theorem.

Step 3. Hence $u \in C^1(J)$,

$$u'(t) = e^{-\rho v(t)} \int_0^t p(s) e^{\rho v(s)} a(s) f(s, u(s)) \,\mathrm{d}s, \quad t \in J. \tag{17}$$

and

$$u''(t) = \rho p(t) u'(t) + a(t) p(t) f(t, u(t)), \quad t \in (0, T].$$

Consequently, u is a solution of (5). Put $t = 0$ in (16) and $t = T$ in (17) and get $u(0) = u'(T)$. Besides, if $a(0) = 0$, then $u'(0) = 0$. As a result u is a solution of problem (5), (6).
Since

$$\begin{aligned} \lim_{n\to\infty} {}^c\!D^{\alpha_{\ell_n}} u_{\ell_n}(t) &= \lim_{n\to\infty} e^{-\rho v(t)} \int_0^t p(s) e^{\rho v(s)} a(s) f(s, u_{\ell_n}(s)) \,\mathrm{d}s \\ &= e^{-\rho v(t)} \int_0^t p(s) e^{\rho v(s)} a(s) f(s, u(s)) \,\mathrm{d}s \quad \text{uniformly on } J \end{aligned}$$

we have $\lim_{n\to\infty} \| {}^c\!D^{\alpha_{\ell_n}} u_{\ell_n} - u' \| = 0$.

EXAMPLE 3.

Let x_0, μ, γ and r, q satisfy the conditions of Example 1. Then for each $n \in \mathbb{N}$, the problem

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^cD^{\alpha_n}u(t) = \frac{\rho}{t^\mu}{}^cD^{\alpha_n}u(t) + \frac{r(t)+q(t)e^{u(t)-x_0}}{t^\gamma},$$

$$\left.\begin{array}{c} u(0) = {}^cD^{\alpha_n}u(t)|_{t=T} \quad \text{if } \gamma = \mu, \\ {}^cD^{\alpha_n}u(t)|_{t=0} = 0,\ u(0) = {}^cD^{\alpha_n}u(t)|_{t=T} \quad \text{if } \gamma \in [0,\mu) \end{array}\right\}$$

has a solution u_n. Theorem 3 guarantees the existence of a subsequence $\{u_{\ell_n}\}$ of $\{u_n\}$ and a solution u to the problem

$$u''(t) = \frac{\rho}{t^\mu}u'(t) + \frac{r(t)+q(t)e^{u(t)-u_0}}{t^\gamma},$$

$$\left.\begin{array}{c} u(0) = u'(T) \quad \text{if } \gamma = \mu, \\ u'(0) = 0,\ u(0) = u'(T) \quad \text{if } \gamma \in [0,\mu) \end{array}\right\}$$

satisfying

$$\lim_{n\to\infty} \|u_{\ell_n} - u\| = 0, \lim_{n\to\infty} \|{}^cD^{\alpha_{\ell_n}}u_{\ell_n} - u'\| = 0.$$

EXAMPLE 4

Let γ, μ and r, q satisfy the conditions of Example 2. Then for each $n \in \mathbb{N}$, the problem

$$\left.\begin{array}{c} \dfrac{\mathrm{d}}{\mathrm{d}t}{}^{c}D^{\alpha_n}u(t) = \dfrac{\rho}{t^{\mu}}{}^{c}D^{\alpha_n}u(t) + r(t) + q(t)|u(t)|^{\gamma}, \\ {}^{c}D^{\alpha_n}u(t)|_{t=0} = 0, \quad u(0) = {}^{c}D^{\alpha_n}u(t)|_{t=T} \end{array}\right\}$$

has a solution u_n and, by Theorem 3, there exists a subsequence $\{u_{\ell_n}\}$ of $\{u_n\}$ and a solution u to the problem

$$\left.\begin{array}{c} u''(t) = \dfrac{\rho}{t^{\mu}}u'(t) + r(t) + q(t)|u(t)|^{\gamma}, \\ u'(0) = 0, \;\; u(0) = u'(T), \end{array}\right\}$$

such that $\lim_{n\to\infty} \|u_{\ell_n} - u\| = 0$, $\lim_{n\to\infty} \|{}^{c}D^{\alpha_{\ell_n}}u_{\ell_n} - u'\| = 0$.

4. Problem with $\rho > 0$

We discuss the sequence of singular fractional Dirichlet problems

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^cD^{\alpha_n}u(t) = \rho p(t){}^cD^{\alpha_n}u(t) + p(t)a(t)f(t, u(t)), \quad n \in \mathbb{N}, \tag{18}$$

$$\left.\begin{array}{c} u(0) = 0, \quad u(T) = 0 \quad \text{if } a(0) > 0, \\ u(0) = 0, \quad u(T) = 0, \quad {}^cD^{\alpha_n}u(t)|_{t=0} = 0 \quad \text{if } a(0) = 0, \quad n \in \mathbb{N}. \end{array}\right\} \tag{19}$$

and singular differential Dirichlet problem

$$u''(t) = \rho p(t)u'(t) + p(t)a(t)f(t, u(t)), \tag{20}$$

$$\left.\begin{array}{c} u(0) = 0, \quad u(T) = 0 \quad \text{if } a(0) > 0, \\ u(0) = 0, \; u(T) = 0, \; u'(0) = 0 \quad \text{if } a(0) = 0. \end{array}\right\} \tag{21}$$

4a. Solvability of fractional BVPs

For $n \in \mathbb{N}$, let $\mathcal{H}_n$ be an operator acting on $C(J)$ defined by the formula

$$(\mathcal{H}_n x)(t) = \frac{\int_0^t (t-s)^{\alpha_n - 1} z(s)\,\mathrm{d}s}{\int_0^T (T-s)^{\alpha_n - 1} z(s)\,\mathrm{d}s} \int_0^T \frac{(T-s)^{\alpha_n - 1}}{\Gamma(\alpha_n)} (\mathcal{K}x)(s)\,\mathrm{d}s - \int_0^t \frac{(t-s)^{\alpha_n - 1}}{\Gamma(\alpha_n)} (\mathcal{K}x)(s)\,\mathrm{d}s,$$

where $\mathcal{K}$ acting on $C(J)$ is defined as

$$(\mathcal{K}x)(t) = \begin{cases} e^{-\rho v(t)} \int_t^T p(s) e^{\rho v(s)} a(s) f(s, x(s))\,\mathrm{d}s & \text{if } t \in (0, T], \\ \dfrac{a(0) f(0, x(0))}{\rho} & \text{if } t = 0, \end{cases}$$

$$z(t) = \begin{cases} e^{-\rho v(t)} & \text{if } t \in (0, T], \\ 0 & \text{if } t = 0, \end{cases} \qquad v(t) = \int_t^T p(s)\,\mathrm{d}s \;\; \text{for } t \in (0, T].$$

By the Riemann-Liouville fractional integral we can write the operator $\mathcal{H}_n$ as

$$(\mathcal{H}_n x)(t) = \frac{I^{\alpha_n}[z](t)}{I^{\alpha_n}[z](T)} I^{\alpha_n}[\mathcal{K}x](T) - I^{\alpha_n}[\mathcal{K}x](t).$$

LEMMA 4.

Let (H_1) hold. Then

(i) $\mathcal{H}_n : C(J) \to C(J)$,

(ii) $\mathcal{H}_n$ is a completely continuous operator,

(iii) u is a solution of problem

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^cD^{\alpha_n}u(t) = \rho p(t){}^cD^{\alpha_n}u(t) + p(t)a(t)f(t,u(t)), \quad n \in \mathbb{N}, \tag{22}$$

$$\left.\begin{array}{c} u(0)=0, \quad u(T)=0 \quad \text{if } a(0)>0, \\ u(0)=0, \quad u(T)=0, \quad {}^cD^{\alpha_n}u(t)|_{t=0}=0 \quad \text{if } a(0)=0, \;\; n\in\mathbb{N}. \end{array}\right\} \tag{23}$$

if and only if it is a fixed point of $\mathcal{H}_n$.

Proof (iii).

Let u be a solution of (22), (23). Then ${}^cD^{\alpha_n}u \in C(J) \cup C^1(0,T]$ and the function $y(t) = {}^cD^{\alpha_n}u(t)$, $t \in J$, is a solution of the differential equation

$$y' = \rho p(t)y + p(t)a(t)f(t,y) \tag{24}$$

in the set $C(J)$. It is not difficult to verify that $\{cz(t) - (\mathcal{K}u)(t) : c \in \mathbb{R}\}$ is the set of all solutions to (24). Hence there exists a unique $c_0 \in \mathbb{R}$ such that

$${}^cD^{\alpha_n}u(t) = c_0 z(t) - (\mathcal{K}u)(t) \ \text{ for } t \in J. \tag{25}$$

If $a(0) = 0$, then it follows from the definition of $\mathcal{K}$ that ${}^cD^{\alpha_n}u(t)|_{t=0} = 0$. We apply I^{α_n} to both sides of (25) and get

$$u(t) = u(0) + c_0 I^{\alpha_n}[z](t) - I^{\alpha_n}[\mathcal{K}u](t) \ \ t \in J.$$

Due to $u(0) = 0$ and $u(T) = 0$ we have $c_0 = I^{\alpha_n}[\mathcal{K}u](T)/I^{\alpha_n}[z](T)$, and therefore

$$u(t) = \frac{I^{\alpha_n}[z](t)}{I^{\alpha_n}[z](T)} I^{\alpha_n}[\mathcal{K}u](T) - I^{\alpha_n}[\mathcal{K}u](t), \quad t \in J.$$

Hence $u = \mathcal{H}_n u$, that is, u is a fixed point of $\mathcal{H}_n$.

Let u be a fixed point of $\mathcal{H}_n$. Then $u \in C(J)$ and $u(t) = (\mathcal{H}_n u)(t)$ for $t \in J$. Therefore $u(0) = 0$, $u(T) = 0$ and applying ${}^cD^{\alpha_n}$ on $u = \mathcal{H}_n u$ we arrive at

$$ {}^cD^{\alpha_n} u(t) = \frac{I^{\alpha_n}[\mathcal{K}u](T)}{I^{\alpha_n}[z](T)} z(t) - (\mathcal{K}u)(t) \quad \text{for } t \in J. \tag{26} $$

Thus ${}^cD^{\alpha_n} u \in C(J)$, and if $a(0) = 0$, then ${}^cD^{\alpha_n} u(t)|_{t=0} = -(\mathcal{K}u)(0) = 0$. Diffentiating of (26)

$$ \frac{\mathrm{d}}{\mathrm{d}t} {}^cD^{\alpha_n} u(t) = \rho p(t) {}^cD^{\alpha_n} u(t) + p(t)a(t)f(t, u(t)), \quad t \in (0, T]. $$

As a result u is a solution of (22), (23).

THEOREM 4 (Existence theorem).
Let (H_1) *hold and let*
(H_3) *for* $t \in J$ *and* $x \in \mathbb{R}$, *the estimate*

$$a(t)|f(t,x)| \le \phi(|x|)$$

is fulfilled, where $\phi \in C[0,\infty)$, ϕ *is nondecreasing and*

$$\lim_{y\to\infty} \frac{\phi(y)}{y} = 0.$$

Then for each $n \in \mathbb{N}$ *there exist a solution* u_n *of problem* (22), (23) *and a positive constant* S *such that*

$$\|u_n\| < S \quad \text{for } n \in \mathbb{N}.$$

Proof. It follows from (H_3) that there exists $S > 0$ such that

$$\frac{\max\{1, T\}\phi(S)}{\rho\Delta}\left(1 + \frac{2}{z(T/2)}\right) < S,$$

where $\Delta = \min\{\Gamma(s) : 1 \le s \le 2\}$. Let

$$\Omega = \{x \in C(J) : \|x\| \le S\}.$$

Choose $n \in \mathbb{N}$. The restriction of $\mathcal{H}_n$ to Ω is a compact operator. In view of

$$a(t)|f(t, x(t))| \le \phi(\|x\|) \le \phi(S), \ \ t \in J, \ \ x \in \Omega$$

we obtain

$$\begin{aligned} |(\mathcal{H}_n x)(t)| &\le \frac{t^{\alpha_n}}{z(T/2)}\left(\frac{2}{T}\right)^{\alpha_n} \cdot \frac{\phi(\|x\|}{\rho} \int_0^T \frac{(T-s)^{\alpha_n}}{\Gamma(\alpha_n)}\,\mathrm{d}s \\ &\quad + \frac{\phi(\|x\|)}{\rho} \int_0^t \frac{(t-s)^{\alpha_n}}{\Gamma(\alpha_n)}\,\mathrm{d}s \\ &\le \frac{\max\{1, T\}\phi(S)}{\rho\Delta}\left(1 + \frac{2}{z(T/2)}\right), \quad t \in J, \ x \in \Omega. \end{aligned}$$

Hence $\|\mathcal{H}_n x\| < S$ for $x \in \Omega$. In particular, $\|\mathcal{H}_n x\| < S$ for $x \in \partial\Omega$, which means that $\mathcal{H}_n$ maps $\partial\Omega$ into Ω. Thus there exists a fixed point u_n of $\mathcal{H}_n$ in Ω. Consequently, $\|u_n\| < S$, and, by Lemma 4, u_n is a solution of (22), (23).

Relations between solutions of fractional and differential problems

THEOREM 5.

Let (H_1) *and* (H_3) *hold. Let* u_n *be a solution of* (22), (23) *satisfying* $\|u_n\| < S$ *for* $n \in \mathbb{N}$, *where* S *is from Theorem 4. Then there exist a subsequence* $\{u_{\ell_n}\}$ *of* $\{u_n\}$ *and a solution* u *of the differential problem* (20), (21) *such that*

$$\lim_{n\to\infty} \|u_{\ell_n} - u\| = 0, \quad \lim_{n\to\infty} \|{}^cD^{\alpha_{\ell_n}} u_{\ell_n} - u'\| = 0. \tag{27}$$

Proof.

- $\{u_n\}$ is bounded in $C(J)$ and it is equicontinuous on J.
- The Arzelà-Ascoli theorem guarantees the existence of a subsequence $\{u_{\ell_n}\}$ of $\{u_n\}$ and $u \in C(J)$ such that $\lim_{n\to\infty} \|u_{\ell_n} - u\| = 0$.
- We prove that u is a solution of (20), (21) and $\lim_{n\to\infty} \|{}^cD^{\alpha_{\ell_n}} u_{\ell_n} - u'\| = 0$.

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