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Streamlining the Mathematics Studies at the Faculty of Science of Palacky University in Olomouc

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Solutions for continuous systems

Quasi periodic and Stepanov case

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Recall of some definitions

- Each a.p. admits a Fourier expansion:

$$f \sim \sum_{\lambda \in \mathbb{R}} a_\lambda e_\lambda$$

  with convergence in quadratic mean.
- Harmonic synthesis in Besicovitch's space:

$$B^2 \sim \left\{ \sum_{\lambda \in \mathbb{R}} a_\lambda e_\lambda, \ \sum_{\lambda \in \mathbb{R}} |a_\lambda|^2 < \infty \right\}$$

- f is quasi-periodic if:

$$\Lambda(f) = \{\lambda \in \mathbb{R}, \quad a_\lambda(f) \neq 0\}$$

  is s.t. its $\mathbb{Z}-$module, $\mathrm{Mod}(f)$, has a finite basis $\omega = (\omega_1, \dots, \omega_m)$.
- Notations QP^0_ω, QP^m_ω, B^2_ω, $B^{m,2}_\omega$.

Fourier Analysis on $L^2(\mathbb{T}^m)$

- $\mathbb{T}^m = (\mathbb{R}/(2\pi\mathbb{Z}))^m$.
- Fourier coefficient:

$$\hat{u}(k) = \int_{\mathbb{T}^m} u(x) e^{-ik.x} \frac{dx}{(2\pi)^m}, \quad k \in \mathbb{Z}^m,$$

- Harmonic synthesis in $L^2(\mathbb{T}^m)$:

$$(u \in L^2(\mathbb{T}^m)) \Leftrightarrow ((\hat{u}(k))_k \in \ell^2(\mathbb{Z}^m))$$

and

$$\|u\|_{L^2}^2 = \sum_{k \in \mathbb{Z}^m} |\hat{u}(k)|^2.$$

Some Hilbert spaces related to multiperiodic case

- Spaces $H^p(\mathbb{T}^m)$ defined by induction:

$$H^p(\mathbb{T}^m) = \left\{ u \in L^2(\mathbb{T}^m),\ Du \in H^{p-1}(\mathbb{T}^m) \right\}$$

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$$\partial_\omega u(x) = Du(x).\omega = \lim_{t \to 0} \frac{u(x + t\omega) - u(x)}{t}$$

- Spaces $H^p_\omega(\mathbb{T}^m)$ defined by induction:

$$H^p_\omega(\mathbb{T}^m) = \left\{ u \in L^2(\mathbb{T}^m),\ \partial_\omega u \in H^{p-1}_\omega(\mathbb{T}^m) \right\}$$

- $H^p_\omega(\mathbb{T}^m)$ is a Hilbert space with the norm:

$$\|u\|^2_{H^p_\omega(\mathbb{T}^m)} = \sum_{j=0}^{p} \|\partial^j_\omega u\|^2_{L^2}.$$

Link with Fourier analysis

For $u \in L^2(\mathbb{T}^m)$, are equivalent:

- $u \in H^1_\omega(\mathbb{T}^m)$
- $\sum_{k \in \mathbb{Z}^m} (k.\omega)^2 |\hat{u}(k)|^2 < \infty$
- There exists $v \in L^2(\mathbb{T}^m)$ s.t. for every $\varphi \in C^1(\mathbb{T}^m)$:

$$\int_{\mathbb{T}^m} v.\varphi = -\int_{\mathbb{T}^m} u.\partial_\omega \varphi$$

- the distribution $\partial_\omega u$ is in $L^2(\mathbb{T}^m)$

and in this case we have:

$$v(x) = \partial_\omega u(x) = \sum_{k \in \mathbb{Z}^m} i(k.\omega)\hat{u}(k)e^{ikx}$$

Small divisors

- $$\inf_{k\in\mathbb{Z}^m\setminus\{0\}} |k.\omega| = 0$$

  (but in $H^1(\mathbb{T}^m)$, $\inf_{k\in\mathbb{Z}^m\setminus\{0\}} |k| = 1 > 0$).

- $$\{t\omega, \quad t \in \mathbb{R}\}$$

  is dense in $\mathbb{T}^m$.

The Operator $\mathcal{Q}_\omega$

- $\mathcal{Q}_\omega(u) := [t \mapsto u(t\omega)]$.
- $\mathcal{Q}_\omega(C^0(\mathbb{T}^m, \mathbb{R}^N)) = QP^0_\omega(\mathbb{R}^N)$, and for every $u \in C^0(\mathbb{T}^m, \mathbb{R}^N)$, we have $\|\mathcal{Q}_\omega(u)\|_\infty = \|u\|_\infty$.
- Let $r \in \mathbb{N}_* \cup \{\infty\}$. Then we have $\mathcal{Q}_\omega(C^r_\omega(\mathbb{T}^m, \mathbb{R}^N)) = QP^r_\omega(\mathbb{R}^N)$, and for every $u \in C^r_\omega(\mathbb{T}^m, \mathbb{R}^N)$, we have $\mathcal{Q}_\omega(Du(\cdot).\omega) = \frac{d}{dt}\mathcal{Q}_\omega(u)$.
- $\mathcal{Q}_\omega(L^2(\mathbb{T}^m, \mathbb{R}^N)) = B^2_\omega(\mathbb{R}^N)$, and for every $u \in L^2(\mathbb{T}^m, \mathbb{R}^N)$, we have $\|\mathcal{Q}_\omega(u)\|_{B^2} = \|u\|_{L^2}$.
- For $j = 1, 2$, $\mathcal{Q}_\omega(H^j_\omega(\mathbb{T}^m, \mathbb{R}^N)) = B^{j,2}_\omega(\mathbb{R}^N)$, and for every $u \in H^j_\omega(\mathbb{T}^m, \mathbb{R}^N)$, we have $\mathcal{Q}_\omega(\partial^j_\omega u) = \nabla^j(\mathcal{Q}_\omega(u))$.

Here P is compact of $\mathbb{R}^M$ either $P = \cup_n K_n$, K_n compact subset of the interior of K_{n+1} for each n.

- Set $QPU^0_\omega = \{f \in APU(\mathbb{R} \times P, \mathbb{R}^N), \quad f(.,\alpha) \in QP^0_\omega(\mathbb{R}, \mathbb{R}^N), \forall \alpha \in P\}$.
- $QPU^0_\omega(\mathbb{R} \times P, \mathbb{R}^N)$, $QP^0_\omega(\mathbb{R}, C^0(P, \mathbb{R}^N))$, $C^0(\mathbb{T}^m, C^0(P, \mathbb{R}^N))$, $C^0(\mathbb{T}^m \times P, \mathbb{R}^N)$ are isomorphic as Fréchet spaces.

Two equations

- We look for $\omega-$q.p. solutions for:

$$q^{(p)}(t) = f(t, q(t), \ldots, q^{(p-1)}(t))$$

- by looking for multiperiodic solutions of the P.D.E.:

$$\partial_\omega^p u(x) = F(x, u(x), \ldots, \partial_\omega^{(p-1)} u(x)),$$

where $F(t\omega, y) = f(t, y)$.

- q is a weak (resp. strong) solution of the first equation iff $\mathcal{Q}_\omega^{-1}(u)$ is a weak (resp. strong) solution of the second one.

A regularization property

- Assume that we have a weak solution $u \in H^p(\mathbb{T}^m, \mathbb{R}^N)$ of:

$$\partial_\omega^p u(x) = F(x, u(x), \dots, \partial_\omega^{(p-1)} u(x)),$$

  and consider $q = \mathcal{Q}_\omega^{-1}(u)$. Then

- q is a weak q.p. of:

$$q^{(p)}(t) = f(t, q(t), \dots, q^{(p-1)}(t))$$

- there exists $\Xi \subset \omega^\perp$ with a complement negligible in $\omega^\perp$ s.t. for any $\xi \in \Xi$, $u(.\omega + \xi) \in C^p \cap B_\omega^{p,2}$ and:

$$\forall t \in \mathbb{R}, \frac{d^p}{dt^p} u(t\omega + \xi) = F\left(t\omega + \xi, u(t\omega + \xi), \dots, \frac{d^{p-1}}{dt^{p-1}} u(t\omega + \xi)\right)$$

Morre concerning the regularization property

If moreover F is bounded of each $\mathbb{R} \times K \times (\mathbb{R}^N)^{p-1}$ where K is any compact subset. Then $q = \mathcal{Q}_\omega^{-1}(u) \in BC^p(\mathbb{R}, \mathbb{R}^N) \cap B^{p,2}(\mathbb{R}^N)$ and satisfies for each $t \in \mathbb{R}$:

$$q^{(p)}(t) = f(t, q(t), \ldots, q^{(p-1)}(t)).$$

Singular perturbations

- The $\omega-$q.p. solutions of :

$$q''(t) = \frac{\partial V}{\partial y}(t, q(t))$$

are searched by the way of the PDE:

$$\partial_\omega^2 u(x) = \frac{\partial A}{\partial y}(x, u(x)),$$

with $A(t\omega, y) = V(t, y)$.

- Set $p > 0$ and introduce the perturbate equation:

$$\partial_\omega^2 u(x) + \frac{1}{p}\left(\Delta u(x) - u(x)\right) = \frac{\partial A}{\partial y}(x, u(x))$$

or:

$$\sum_{1 \le j,k \le m} a_{ij}^p \frac{\partial^2 u}{\partial x_j \partial x_k}(x) - \frac{1}{p} u(x) = \frac{\partial A}{\partial y}(x, u(x))$$

with $a_{ij}^p = \omega_i \omega_j + \delta_{ij}/p$.

Assumptions on A

(A1) A is measurable, and for all $x \in \mathbb{T}^N$, the function $A(x,.) : \mathbb{T}^N \to \mathbb{R}$ is of class C^1 and convex,

(A2) There exists $\varphi_0 \in L^2$ s.t. $A(.,\varphi_0(.)) \in L^1(\mathbb{T}^N;\mathbb{H})$ and $\frac{\partial A}{\partial y}(.,\varphi_0(.)) \in L^2$,

(A3) $\exists a \in L^2, \quad \exists b \in L^1(\mathbb{T}^N;\mathbb{R}), \quad A(x,y) \geq < a(x), y > + b(x)$,

(A4) $\exists(\alpha_1,\beta_1,\gamma_1) \in \mathbb{R}_*^+ \times L^2(\mathbb{T}^N;\mathbb{R}) \times L^1(\mathbb{T}^N;\mathbb{R})$, $\forall (x,y) \in \mathbb{T}^N \times \mathbb{H}$:

$$\left\langle \frac{\partial A(x,y)}{\partial y}, y \right\rangle \geq \alpha_1 |y|^2 - \beta_1(x)|y| - \gamma_1(x),$$

and $\|\beta_1\|_{L^2}^2 + 4\min\{1,\alpha_1\}\left(\int_{\mathbb{T}^N} \gamma_1 d\mu\right) \geq 0$,

(A5) $\exists(c,d) \in L^2(\mathbb{T}^N;\mathbb{R}) \times L^1(\mathbb{T}^N,\mathbb{R})$:

$$\left| \frac{\partial A}{\partial y}(x,y) \right| \leq c(x)|y| + d(x) \qquad a.e.$$

Example of V

- Equation is:

$$q''(t) = \varphi(t)f'(q(t)) + \psi(t).$$

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$$V(t, y) = \varphi(t)f(y) + \psi(t)y.$$

- $\varphi, \psi \in AP^0(\mathbb{R}, \mathbb{R})$ and $\inf \varphi > 0$.
- $f \in C^1(\mathbb{R}, \mathbb{H})$ is convex.
- $\exists(\alpha, \beta) \in \mathbb{R}_*^+ \times \mathbb{R}$, $y \mapsto < f'(y), y > -\alpha|y|^2 + \beta|y|$ is bounded from below.
- $\exists(\mu, \nu) \in (\mathbb{R}_*^+)^2$, $\forall y \in \mathbb{R}$, $|f'(y)| \leq \mu|y| + \nu$.

Potential

- Our equation gives the critical points of the functional:

$$\phi_p(u) = \frac{1}{2}\sum_{j,k} a^p_{jk}\left(\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_k}\right)_{L^2} + \frac{1}{2p}\|u\|^2_{L^2} + \int_{\mathbb{T}^N} A(x, u(x))\frac{dx}{(2\pi)^m}.$$

- $\phi_p : H^1(\mathbb{T}^m) \to [0; \infty]$ is a l.s.c. convex and coercive functional.
- it admits a minimum $u_p \in H^1(\mathbb{T}^m)$ which satisfies $0 \in \partial\phi_p(u_p)$.
- u_p is a solution of the perturbative equation.
- $(u_p)_p$ is bounded in $H^1_\omega(\mathbb{T}^m)$, so has a weak limit U.
- U is a weak solution of $\partial^2_\omega U(x) = \frac{\partial A}{\partial y}(x, U(x))$.
- $q(t) = U(t\omega)$ is a weak solution of the first problem

Almost periodic case

- $$q''(t) = F'(q(t)) + b(t)$$
- $b \in AP^0$, $F \in C^1$ is convex and:

  $$< F'(y), y > \geq \alpha |y|^2 - \beta |y| - \gamma,$$

  with $\alpha > 0$ and $\beta^2 + 4\gamma \min\{1, \alpha\}$.
- Then there exists an a.p. solution.

Stepanov a.p. functions

$f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}, E)$, $p \geq 1$, is Stepanov-a.p. ($\mathrm{S}^p_{\mathrm{ap}}$) if:

$$\forall \varepsilon > 0, \exists \ell > 0, \forall a \in \mathbb{R}, \exists \tau \in [a, a+\ell],$$

$$\sup_{x \in \mathbb{R}} \left[\int_x^{x+1} \left| f(t+\tau) - f(t) \right|_E^p \, dt \right]^{\frac{1}{p}} < \varepsilon.$$

- Assume that $p^{-1} + q^{-1} = r^{-1}$ with $p, q, r \in [1, \infty]$. Then:
- if $A \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, \mathcal{L}(E))$ and $\beta \in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R}, E)$, then $A.\beta \in \mathrm{S}^r_{\mathrm{ap}}(\mathbb{R}, E)$,
- if $\alpha \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, \mathbb{R})$ and $\beta \in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R}, E)$, then $\alpha.\beta \in \mathrm{S}^r_{\mathrm{ap}}(\mathbb{R}, E)$.
- if $A \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, \mathcal{L}(E)) \cap \mathrm{L}^\infty(\mathbb{R}, \mathcal{L}(E))$ and $\beta \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E) \cap \mathrm{L}^\infty(\mathbb{R}, E)$, then $A.\beta \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E)$.
- if $\alpha \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, \mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R}, \mathbb{R})$ and $\beta \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E) \cap \mathrm{L}^\infty(\mathbb{R}, E)$, then $\alpha.\beta \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E)$.

If $f \in \mathrm{C}^0(E, E)$, $a, b > 0$ and $p, q \geq 1$ satisfy:

$$\forall x \in E,\ |f(x)|_E \leq a|x|_E^{p/q} + b,$$

then, for any $g \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E)$, we have: $f \circ g \in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R}, E)$ (i.e. the autonomous Nemytskii operator $\mathcal{N}_f$ maps $\mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E)$ in $\mathrm{S}^q_{\mathrm{ap}}(\mathbb{R}, E)$).

Assume that:

- for all $t \in \mathbb{R}$, $f(t, .) \in L^{\infty}(E, E)$;
- $[t \mapsto f(t, .)] \in \mathrm{S}^{q}_{\mathrm{ap}}(\mathbb{R}, L^{\infty}(E, E))$.

Then $\mathcal{N}_f := [u \mapsto [t \mapsto f(t, u(t))]]$ maps $\mathrm{S}^{p}_{\mathrm{ap}}(\mathbb{R}, E)$ into $\mathrm{S}^{q}_{\mathrm{ap}}(\mathbb{R}, E)$, where $p, q \geq 1$.

Set $F(t,X) := A(t).F_1(X)$, $p,q \geq 1$, $r \in (0, p/q]$, where:

- $A \in \mathrm{S}^{\frac{pq}{p-qr}}(\mathbb{R}, \mathcal{L}(E))$;
- $F_1 \in C^0(E,E)$ and satisfies:

$$\forall X \in E, \quad |F_1(X)|_E \leq a|X|_E^r + b,$$

with $a, b > 0$.

Then $\mathcal{N}_F : \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E) \to \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R}, E)$.

The main property.

- Let us consider $f \in AC_{loc}(\mathbb{R}, E)$ s.t. f and f' are bounded in the Stepanov norm
- Then f is bounded in the essential sup-norm.
- As a consequence, if $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, E)$, and f and f' are Stepanov a.p., then f is Bohr (uniformly) almost-periodic.

Consequence of the main property.

Assume that $\mathcal{N}_F \colon \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E) \to \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R}, E)$, where $p, q \geq 1$. Then every $\mathrm{S}^p_{\mathrm{ap}}$-solution of the differential equation $X' = F(t, X)$, in a uniformly convex Banach space E, is uniformly almost-periodic.

Application 1.

Assume that there exist constants p, q, r with $p, q \geq 1$, $r^{-1} = p^{-1} + q^{-1}$ and $r \in (0, \frac{p}{q}]$, such that the following conditions are satisfied:

- $A \in \mathrm{S}_{\mathrm{ap}}^{\frac{pq}{p-qr}}(\mathbb{R}, \mathcal{L}(E))$;
- $\forall X \in E, \ \ |F_1(X)|_E \leq C_1|X|_E^r + C_2$, with $C_1, C_2 \geq 0$, holds for $F_1 \in \mathrm{C}^0(E, E)$;
- for all $t \in \mathbb{R}, \ \ F_2(t, .) \in \mathrm{L}^\infty(E, E)$;
- $[t \mapsto F_2(t, .)] \in \mathrm{S}_{\mathrm{ap}}^q(\mathbb{R}, \mathrm{L}^\infty(E, E))$.

Then the equation $X' = A(t).F_1(X) + F_2(t, X)$ has no purely $\mathrm{S}_{\mathrm{ap}}^p$-solution.

Application 2.

Assume that there exist constants $\alpha > 1$, $r > 0$, such that:

- $A \in \mathrm{S}^{\alpha}_{\mathrm{ap}}(\mathbb{R}, \mathcal{L}(E))$;
- $F_1 \in \mathrm{C}^0(E, E)$ and satisfies:

$$\forall X \in E, \quad |F_1(X)|_E \leq C_1 |X|^r + C_2,$$

with $C_1, C_2 \geq 0$.

Then, for any $p > \frac{\alpha r}{\alpha - 1}$, the equation $X' = A(t).F_1(X)$ has no purely $\mathrm{S}^p_{\mathrm{ap}}$-solution. In particular, for $r < 1 - \alpha^{-1}$, there are no purely S_{ap}-solutions.