INVESTMENTS IN EDUCATION DEVELOPMENT

Streamlining the Mathematics Studies at the Faculty of Science of Palacky University in Olomouc

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Introduction to almost periodicity

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Introduction

3 subjects:

- introduction to almost periodicity and semi periodicity
- some subjects concerning solutions to continuous equations.
- some subjects concerning solutions to discrete equations.

Motivations

- superposition of periodic motions:

$$t \mapsto \cos(t) + \cos(\alpha t)$$

  (not periodic if $\alpha \notin \mathbb{Q}$).
- link with boundedness: each bounded solution of an autonomous linear system $x' = Ax$ or $x_{t+1} = Ax_t$ is a sum of periodic functions
- motivations from celestial mechanics, routes to chaos, economics.

Definitions

- functions with values in E, a Banach space, mostly $\mathbb{R}^N$ either Hilbert.
- trigonometric polynomial:
  $\sum_{\lambda \in J} a_\lambda e_\lambda, \quad e_\lambda(t) = e^{i\lambda t}, \quad J \subset \mathbb{R} \text{ finite}, \quad a_\lambda \in E.$
- If $Per(\mathbb{R}, E)$ is the set of all continuous periodic functions, f is almost periodic $f \in AP^0(\mathbb{R}, E)$ iff f is uniform limit of functions in the linear space generated by $Per(\mathbb{R}, E)$.

Equivalents Definitions

- f is a.p. iff there exists a sequence of t.p. $(P_n)_n$ s.t.:

$$\|f - P_n\|_\infty \to 0,$$

- f is a.p. iff for all $\forall \varepsilon > 0$, $\exists \ell > 0$, $\forall a \in \mathbb{R}$, $\exists \tau \in [a, a + \ell]$, $\|f(. + \tau) - f(.)\|_\infty \leq \varepsilon$,
- f is a.p. iff for each sequence $(h_n)_n$, there exists a uniformly convergent subsequence for $(f(. + h_n))_n$.
- f is a.p. iff there exists $\varphi \in C^0(b\mathbb{R})$ s.t. $\varphi \circ in = f$.

Some variants

- former definitions w.r.t. the sup norm: Bohr definition
- S^p-Stepanov ($f \in L^p_{loc}$):

$$\|f\| = \left(\sup_{a \in \mathbb{R}} \int_a^{a+1} |f(t)|^p dt \right)^{1/p}.$$

  Stepanov a.p. functions with values in E can be seen as Bohr a.p. functions with values in $L^p([0;1], E)$ via $t \mapsto [u \mapsto f(t+u)]$.
- B^p-Besicovitch (in the quotient space in order to make it a norm):

$$\|f\| = \limsup_{T \to \infty} \left(\frac{1}{2T} \int_{-T}^{T} |f(t)|^p dt \right)^{1/p}.$$

- others : Weyl, equi-Weyl.
- Paper by Andres, Bersani and Grande.
- also some variants using another (topological) groups than $\mathbb{R}$, e.g. $\mathbb{Z}$.

Fourier Analysis

- mean operator (formula non depending on the period for periodic functions):

$$\mathcal{M}\{f\} = \mathcal{M}\{f(t)\}_t = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt$$

  and Fourier coefficients: $a_\lambda(f) = \mathcal{M}\{fe_{-\lambda}\}$

- each a.p. admits a Fourier expansion:

$$f \sim \sum_{\lambda\in\mathbb{R}} a_\lambda(f) e_\lambda$$

  with convergence in quadratic mean.

- Parseval's relation:

$$\mathcal{M}\{|f|^2\} = \sum_{\lambda\in\mathbb{R}} |a_\lambda(f)|^2 < \infty.$$

- Harmonic synthesis in Besicovitch's space:

$$B^2 = \left\{ \sum a_\lambda e_\lambda, \ \sum |a_\lambda|^2 < \infty \right\}$$

Blot's spaces

- ∇ infinitesimal generator of the group of translations in B^2:

$$\nabla f \sim \lim_{s \to O} \frac{f(. + s) - f(.)}{s}.$$

- Domain:

$$B^{1,2} = \left\{ f \in B^2, \quad \nabla f \in B^2 \right\}.$$

- By induction: $B^{m,2}$, which is an Hilbert space when equipped with the norm:

$$\|u\|^2_{B^{m,2}} = \sum_{j=0}^{m} \|\nabla^j u\|^2_{L^2}.$$

- $f \in B^{1,2}$ iff $\sum_\lambda (1 + |\lambda|^2)|a_\lambda(f)|^2 < \infty$, and:

$$\nabla f = \sum_{\lambda \in \mathbb{R}} i\lambda a_\lambda(f) e_\lambda.$$

Bohl-Bohr property

- Let f be a.p. Then its indefinite integrals are a.p. iff they are bounded.
- $\mathcal{M}\{f\} = 0$ does not imply that $\int f$ is a.p.

$$f(t) = \sum_{n\geq 1} \frac{1}{n^{3/2}} e^{\frac{it}{n}}$$

- If f is Stepanov a.p., then its definite integrals are (Bohr)a.p. iff they are bounded in the Stepanov norm.

Nemytskii Operators

- $APU(\mathbb{R} \times E, F)$ is the set of continuous functions s.t. for each compact subset K of E and for all $\varepsilon > 0$, $\exists \ell > 0$, $\forall a \in \mathbb{R}$, $\exists \tau \in [a, a+\ell]$, $\sup_{(x,t)\in K\times\mathbb{R}} \|f(t+\tau, x) - f(t,x)\| \leq \varepsilon$
- $F \in APU(\mathbb{R} \times E, F)$ iff its Nemytskii operators maps continuously $AP^0(E)$ in $AP^0(F)$
- if $F \in APU(\mathbb{R} \times E, F)$ and $D_x F \in APU(\mathbb{R} \times E, \mathcal{L}(E, F))$, then $\mathcal{N}_F$ is differentiable from $AP^0(E)$ to $AP^0(F)$ with:

$$\mathcal{N}_F'(u).v = [t \mapsto D_x F(t, u(t)).v(t)].$$

Example: an a.p. optimal control problem

- $f_0 : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ is continuous and $f_0(t, ., .)$ is concave for each t;
- $f(t, x, u) = A(t)x + B(t)u + b(t)$ with A, B, b continuous.
- Problem: maximize $\mathcal{M}\{f_0(t, x(t), u(t)\}_t$ with $x'(t) = f(t, x(t), u(t))$.
- if $f_0(., x, u)$ is $T-$periodic and if there exists (x, u) an a.p. solution, then there exists a $T-$periodic solution
- if f_0 is autonomous and if there exists (x, u) an a.p. solution, there exists a constant one.

Some subclasses

- Let us introduce:

$$\Lambda(f) = \{\lambda \in \mathbb{R}, \quad a_\lambda(f) \neq 0\}$$

  and the $\mathbb{Z}-$module it generates, $\mathrm{Mod}(f)$.

- f is quasi-periodic iff $\mathrm{Mod}(f)$ has a $\mathbb{Z}-$finite basis $\omega = (\omega_1, \ldots, \omega_m)$:.

$$\mathrm{Mod}(f) = \mathbb{Z}\omega_1 + \ldots + \mathbb{Z}\omega_m.$$

- f is semi-periodic iff $\Lambda(f) \subset \theta\mathbb{Q}$.

Semi periodic functions and sequences

- For functions:

$$\forall \varepsilon > 0, \exists T > 0, \forall n \in \mathbb{Z}, \forall x \in \mathbb{R}, \quad |f(x+nT) - f(x)| \leq \varepsilon.$$

- For sequences:

$$\forall \varepsilon > 0, \exists T \in \mathbb{N}, \forall n \in \mathbb{Z}, \forall k \in \mathbb{Z}, \quad |x_{k+nT} - x_k| \leq \varepsilon.$$

- This is a Banach space in case of sequences (Berg, Wilansky), different from the space of a.p. sequences. It is not linear in case of functions, recall that:

$$AP^0(\mathbb{R}, E) = \overline{Vect(\mathrm{Per}(\mathbb{R}, E))}.$$

- The set of s.p. functions (resp. sequences) is the closure of the space of periodic functions (resp. sequences)

Link between s.p. functions and sequences

Given $\underline{x} = (x_t)_t$, and $f_{\underline{x}}$ its piecewise linear interpolation:

$$f_{\underline{x}}(t+u) = (1-u)x_t + ux_{t+1}, \quad (t,u) \in \mathbb{Z} \times [0;1],$$

then the following are equivalent:

- $f_{\underline{x}}$ is s.p. with a semi-period in $\mathbb{N}$.
- There exists a s.p. function with a semi-period in $\mathbb{N}$ whose restriction to $\mathbb{Z}$ is $\underline{x}$.
- $\underline{x}$ is s.p.

Fourier analysis for s.p. functions

- Let f be a.p. Then it is s.p. iff $\Lambda(f) \subset \theta\mathbb{Q}$.
- If T is a semi-period of f and $(f_n)_n$ is a sequence of periodic functions s.t. $\|f_n - f\|_\infty \to 0$, then for sufficiently large n, $T_n/T \in \mathbb{Q}$.
- If f is s.p. and q.p., then f is periodic.

Uniformly s.p. functions with respect to a parameter

- $f : \mathbb{R} \times M \to \mathbb{R}^k$ ($M \subset \mathbb{R}^n$) is u.s.p. if for any compact set $K \subset M \subset \mathbb{R}^n$, we have:

$$\forall \varepsilon > 0, \exists T > 0, \forall n \in \mathbb{Z}, \forall t \in \mathbb{R}, \forall \alpha \in K,$$

$$|f(t + nT, \alpha) - f(t, \alpha)|_{\mathbb{R}^k} \leq \varepsilon.$$

- Any u.s.p. function is a uniform limit, on each $\mathbb{R} \times K$, of a sequence of continuous functions which are periodic w.r.t. their first variables.
- analogous definition and properties in discrete case.

A discrete equation

- Assume that A has no eigenvalues with imaginary part and that f is u.s.p., Lipschitzian w.r.t its second variable with a sufficently small Lispchitz constant. Then the equation:

$$x_{t+1} + Ax_t = f(t, x_t)$$

  has a s.p. solution.
- Main tool: since $\mathcal{N}_f$ maps the set of s.p. sequences into itself, it is possible to look $(x_t)_t$ as a fixed point of $T^{-1} \circ \mathcal{N}_f$, where $T : (x_t) \mapsto x_{t+1} + Ax_t$ is bijective in the set of s.p. sequences with assumptions on A.

A continuous equation

- Assume that A has no eigenvalues on the imaginary axe and that f is u.s.p., Lipschitzian w.r.t its second variable with a sufficently small Lispchitz constant. Then the equation:

$$x' + Ax = f(t, x(t))$$

  has a s.p. solution.
- Here we approximate f by a sequences of periodic functions f_p.