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Streamlining the Mathematics Studies at the Faculty of Science of Palacky University in Olomouc

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(CZ.1.07/2.2.00/28.0141)

ℓ–stable functions

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Outline

Classical differentiability

Theorem 1

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^2 and $x \in \mathbb{R}^n$. If $f'(x) = 0$ and for every $h \in S_{\mathbb{R}^n}$ it holds

$$f^2(x; h, h) := \lim_{t \to 0} \frac{f'(x + th)h - f'(x)h}{t} > 0,$$

then the function f has an isolated minimizer of second-order at x.

Definition

Let X be a normed linear space. We say that a point $x \in X$ is **an isolated minimizer of second-order for a function** $f : X \to \mathbb{R}$ *if there exist a neigbourhood U of x and a constant $K > 0$ such that*

$$f(y) - f(x) \geq K\|y - x\|^2, \quad \forall y \in U.$$

Example 1: For the function $f : \mathbb{R} \to \mathbb{R} : f(x) = x^2$, the point 0 is an isolated minimizer of second-order in contrast to the function $f : \mathbb{R} \to \mathbb{R} : f(x) = x^4$.

Different types of differentiability

Definition G

Let X and Y are normed linear spaces, $f : X \to Y$ and $x \in X$. We say that a function f is **Gâteaux differentiable at** x *if there exists $f'_G(x) \in \mathcal{L}(X, Y)$ such that*

$$f'_G(x)h = \lim_{t \downarrow 0} \frac{f(x + th) - f(x)}{t},$$

for every $h \in X$.

Different types of differentiability

Definition F

Let X and Y are normed linear spaces, $f : X \to Y$ and $x \in X$. We say that a function f is **Fréchet differentiable at** x *if there exists $f'_F(x) \in \mathcal{L}(X, Y)$ such that*

$$f'(x)h = \lim_{t \downarrow 0} \frac{f(x+th) - f(x)}{t},$$

for every $h \in X$ and this limit is uniform for $h \in S_X$.

Different types of differentiability

Definition S

Let X and Y are normed linear spaces, $f : X \to Y$ and $x \in X$. We say that a function f is **strictly differentiable at** x *if there exists $f'_S(x) \in \mathcal{L}(X, Y)$ such that*

$$f'_S(x)h = \lim_{y \to x, t \downarrow 0} \frac{f(y+th) - f(y)}{t},$$

for every $h \in X$ and this limit is uniform for $h \in S_X$.

Different types of differentiability

Theorem 2

Let X and Y are normed linear spaces, $f : X \to Y$ and $x \in X$. If the function $y \mapsto f'_G(y)$ is continuous on some neighbourhood of x, then the function f is strictly differentiable at x.

Since 1980 - efforts to generalize Theorem 1

[1] Ben-Tal, A., Zowe, J.: *Directional derivatives in nonsmooth optimization*, J. Optim. Theory Appl. **47**, 483–490 (1985).

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Special attention – $C^{1,1}$ functions

Definition

Let X, Y be normed linear spaces. We say that a function $f : X \to Y$ is a **$C^{1,1}$ function near $x \in X$** if it is (Fréchet) differentiable on some neighbourhood of x and its derivative $f'(.)$ is Lipschitz there.

Special attention – $C^{1,1}$ functions

$C^{1,1}$ functions are interesting theoretically and they also appear in some problems of applied mathematics:

- the augmented Lagrange methods
- the proximal point method
- the penalty function method

Prologue was in 2006

Ginchev I., Guerraggio A., Rocca M.: *From scalar to vector optimization*. Applications of Mathematics **51** (2006), 5—36.

Theorem 3

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^{1,1}$ function near $x \in \mathbb{R}^n$. If $f'(x) = 0$ and for every $h \in S_{\mathbb{R}^n}$ it holds

$$f_P'^{\ell}(x;h) := \liminf_{t \downarrow 0} \frac{f(x+th) - f(x) - tf'(x;h)}{t^2/2} > 0,$$

then the function f has an isolated minimizer of second-order at x.

Peano derivative of second-order

Peano G.: *Sulla formula di Taylor*. Atti del' Accademia delle Science di Torino **51** (1891), 40—46.

Giuseppe Peano (1858–1932)

Dini derivative of second order

$$f_D''^{\ell}(x; h) := \liminf_{t \downarrow 0} \frac{f'(x+th)h - f'(x)h}{t}$$

Ulisse Dini (1845–1918)

Comparison of Peano and Dini derivative

Torre D.L., Rocca M.: *Remarks on second-order generalized derivatives for differentiable functions with Lipschitzian jacobian*. Applied Mathematics E-Notes **3** (2003), 130—137.

For a function $f : X \to \mathbb{R}$, $x \in X$, $h \in X$ it holds

$$f_P'^{\ell}(x;h) \geq f_D'^{\ell}(x;h)$$

Comparison of Peano and Dini derivative

Example 2: Let us consider a function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \int_0^{|x|} t(\frac{19}{20} + \sin \ln t)dt & , \quad \text{if } x \neq 0, \\ 0 & , \quad \text{if } x = 0. \end{cases}$$

Then for $x \neq 0$:

$$f'(x) = x((19/20) + \sin(\ln |x|)),$$

and

$$f'(0) = 0.$$

Concurrently

$$f_P^{\prime\ell}(0;1) = f_P^{\prime\ell}(0;-1) = \frac{19}{20} + \frac{2}{5}(-\sqrt{5}) > 0,$$

but

$$f_D^{\prime\ell}(0;1) = (19/20) - 1 < 0.$$

Question

Theorem 3

[GGR] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^{1,1}$ function near $x \in \mathbb{R}^n$. If $f'(x) = 0$ and for every $h \in S_{\mathbb{R}^n}$ it holds*

$$f_P^{\prime\ell}(x; h) := \liminf_{t \downarrow 0} \frac{f(x+th) - f(x) - tf'(x;h)}{t^2/2} > 0,$$

then the function f has an isolated minimizer of second-order at x.

?It is possible to weaken regularity in Theorem 3?

ℓ-stable scalar functions

Bednařík D., Pastor K.: *On second-order conditions in unconstrained optimization*. Math. Program. (Ser. A) **113** (2) (2008), 283–298.

Definition

For $f : X \to \mathbb{R}$, $x \in X$, $h \in X$, we define

$$f'(x; h) = \lim_{t \downarrow 0} \frac{f(x + th) - f(x)}{t}.$$

$$f^{\ell}(x; h) = \liminf_{t \downarrow 0} \frac{f(x + th) - f(x)}{t}.$$

Definition

We say that a function $f : X \to \mathbb{R}$ is **ℓ–stable at $x \in X$** if there are a neighbourhood U of x and a constant $K > 0$ such that

$$|f^\ell(y;h) - f^\ell(x;h)| \leq K\|y - x\|, \quad \forall y \in U, \forall h \in S_X.$$

Answer for question

Theorem 4

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an ℓ–stable function at $x \in \mathbb{R}^n$. If $f'(x) = 0$ and for every $h \in S_{\mathbb{R}^n}$ it holds

$$f_P^{\prime\ell}(x; h) = \liminf_{t \downarrow 0} \frac{f(x+th) - f(x) - tf'(x; h)}{t^2/2} > 0,$$

then the function f has an isolated minimizer of second-order at x.

Example 3

Dvorská M.: *Vector optimization*, Diploma thesis, Olomouc (2011).

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$,

$$f(x_1, x_2) = \int_0^{|x_1|} \varphi(u)du,$$

where function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}$ is defined as follows:

$$\varphi(u) = \begin{cases} 1 & \text{if } u > 1, \\ 2u - \frac{1}{n+1} & \text{if } u \in \left(\frac{1}{n+1}, \frac{1}{n}\right],\ n \in \mathbb{N}, \\ 0 & \text{if } u = 0. \end{cases}$$

Example 3

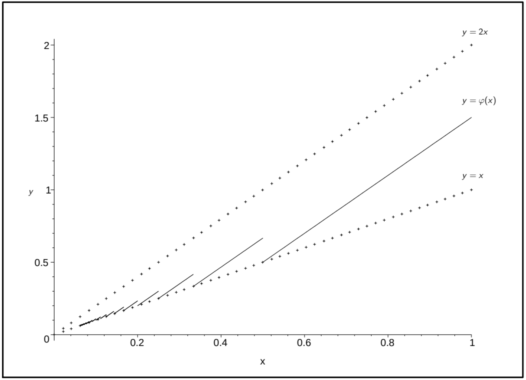


Figure: Graph of function φ on $[0, 1]$

Example 3

The first-order directional derivatives of function f at points

$$a_n = (\frac{1}{n}, 0),\ \ n \in \mathbb{N},\ \ n > 1,$$

in directions

$$\bar{d} = (1, 0),\ \ \hat{d} = (-1, 0)$$

are

$$f'(a_n; \bar{d}) = \frac{1}{n}, \qquad f'(a_n; \hat{d}) = -\frac{n+2}{n(n+1)}.$$

Hence, f is not differentiable on any neighbourhood of point $x_0 = (0, 0)$.

Example 3

$$\left|\liminf_{t\downarrow 0}\frac{f(y+tv)-f(y)}{t}\right| = \begin{cases} |v_1|\left(2|y_1|-\dfrac{1}{n+1}\right) & \text{if } |y_1|\in\left(\frac{1}{n+1},\frac{1}{n}\right), \\ v_1y_1 & \text{if } |y_1|=\frac{1}{n},\ v_1y_1\geq 0, \\ |v_1|\left(2|y_1|-\dfrac{1}{n+1}\right) & \text{if } |y_1|=\frac{1}{n},\ v_1y_1<0, \\ 0 & \text{if } y_1=0. \end{cases}$$

The function f is ℓ-stable at x_0 because:

$$|f^\ell(x_0;v)-f^\ell(y;v)| = |\liminf_{t\downarrow 0}\frac{f(y+tv)-f(y)}{t}| \leq 2\|y\|,$$

$$\forall y\in\mathbb{R}^2,\ \|y\|<1,\ \forall v\in S_{\mathbb{R}^2}.$$

Smoothness properties

Theorem 5

If a function $f : \mathbb{R}^n \to \mathbb{R}$ is ℓ–stable function at $x \in \mathbb{R}^n$, then the function f is continuous on some neighbourhood of x.

Theorem 6

Let X be a normed linear space and $x \in X$. If a function $f : X \to \mathbb{R}$ is ℓ–stable at $x \in X$ and continuous on some neighbourhood of x, then f is Lipschitz on some neighbourhoud of x and it is strictly differentiable at x.

Finite-dimensional characterization 1

Theorem 7

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $x \in \mathbb{R}^n$. Then f is ℓ–stable at x if and only if f is strictly differentiable at x and there exist a neighbourhood U of x and a constant $K > 0$ such that

$$\|f'(y) - f'_S(x)\| \leq K\|y - x\|,$$

for almost all $y \in U$ (in the Lebesque sense).

Finite-dimensional characterization 2

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ *be Lipschitz near* $x \in \mathbb{R}^n$ *and let* $h \in \mathbb{R}^n$*. The* **Clarke upper generalized directional derivative of** f **at** x *is defined by*

$$f^\circ(x; h) = \limsup_{t \downarrow 0} \frac{f(y + th) - f(y)}{t},$$

and the **Clarke generalized gradient of** f **at** x *is defined by*

$$\partial_c f(x) = \{x^* \in \mathbb{R}^n ; \langle x^*, h\rangle \leq f^\circ(x; h), \forall h \in \mathbb{R}^n\}.$$

Finite-dimensional characterization 2

Definition

A set-valued mapping $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ *is* **calm at** x *if* $F(x) \neq \emptyset$ *and there exist a neighbourhood* U *of* x *and a constant* $K > 0$ *such that*

$$F(y) \subset F(x) + K\|y - x\|B_{\mathbb{R}^n}, \quad \forall y \in U.$$

$B_{\mathbb{R}^n} := \{z \in \mathbb{R}^n; \|z\| \leq 1\}$;
for $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^n$ we define $A + B = \{a + b; a \in A, b \in B\}$

Finite-dimensional characterization 2

Theorem 8

Let $f : \mathbb{R}^n \to \mathbb{R}$ is a function and $x \in \mathbb{R}$. Then f is ℓ–stable at x if and only if f is strictly differentiable at x and the mapping $\partial_c f : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is calm at x.

ℓ-stable vector functions

Definition

For $f : X \to Y, x \in X, h \in X$ and $\xi \in Y^*$, we define

$$f_\xi^\ell(x;h) = \liminf_{t \downarrow 0} \frac{\langle \xi, f(x+th) - f(x) \rangle}{t}$$

Definition

Let X and Y be normed linear spaces and $x \in X$. We say that a function $f : X \to Y$ is ℓ**–stable at** $x \in X$ *if there exist a neighbourhood U of x and a constant $K > 0$ such that it holds that*

$$\left| f_\xi^\ell(y;h) - f_\xi^\ell(x;h) \right| \le K \|\xi\| \, \|y - x\| \, , \ \forall y \in U, \forall h \in S_X, \forall \xi \in Y^*.$$

Finite-dimensional characterization 1

Theorem 9

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function and $x \in \mathbb{R}^n$. Then f is ℓ–stable at x if and only if for any $\xi \in \mathbb{R}^m$ the scalar function

$$f_\xi(\cdot) = \langle \xi, f(\cdot) \rangle$$

is ℓ–stable at x.

Finite-dimensional characterization 2

Theorem 10

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function and $x \in \mathbb{R}^n$. Then f is ℓ–stable at x if and only if f is strictly differentiable at x and there exist a neighbourhood U of x and a constant $K > 0$ such that

$$\|f'(y) - f'_S(x)\| \leq K\|y - x\|,$$

for almost all $y \in U$ (in the Lebesque sense).

Finite-dimensional characterization 3

A set $C \subset \mathbb{R}^n$ is called **a cone** if

$$x \in C, \alpha \geq 0 \;\Rightarrow \alpha x \in C.$$

A cone $C \subset \mathbb{R}^n$ is called **pointed** if

$$C \cap (-C) = \{0_{\mathbb{R}^n}\}.$$

For arbitrary cone $C \subset \mathbb{R}^n$, we define **a positive polar cone C^*** and a set $\boldsymbol{\Gamma_C}$:

$$C^* := \{\xi \in \mathbb{R}^n ; \langle \xi, y\rangle \geq 0,\; y \in C\}, \qquad \Gamma_C := C^* \cap S_{\mathbb{R}^n}.$$

Finite-dimensional characterization 3

Theorem 11

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function, $x \in \mathbb{R}^n$, and $C \subset \mathbb{R}^m$ be a closed, convex and pointed cone with non-empty interior. Then f is ℓ–stable at x if and only if there exist a neighbourhood U of x and a constant $K > 0$ such that

$$|f_\xi^\ell(y;h) - f_\xi^\ell(x;h)| \le K\|y - x_0\|, \quad \forall y \in U, \forall h \in S_{\mathbb{R}^n}, \forall \xi \in \Gamma_C.$$

Infinite-dimensional characterization

For $f : X \to Y, x \in X, h \in X$ and $\xi \in Y^*$, we define

$$f_\xi^u(x;h) = \limsup_{t \downarrow 0} \frac{\langle \xi, f(x+th) - f(x) \rangle}{t}$$

Definition

Let X and Y be normed linear spaces and $x \in X$. Then a function $f : X \to Y$ is ℓ–stable at $x \in X$ if and only if there exist a neighbourhood U of x and a constant $K > 0$ such that it holds that

$$\left| f_\xi^u(y;h) - f_\xi^u(x;h) \right| \leq K \|\xi\| \|y - x\|, \ \forall y \in U, \forall h \in S_X, \forall \xi \in Y^*.$$

Smoothness properties

Theorem 12

Let X be a normed linear space, Y be a Banach space, and $f : X \to Y$ be a continuous function near $x \in X$. If f is an ℓ–stable function at x, then f is strictly differentiable at x.

Vector optimization – e-minimizer

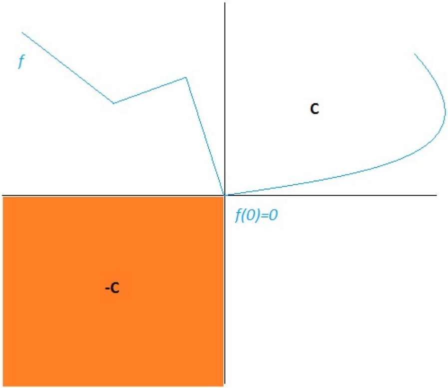
Definition

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function, $C \subset \mathbb{R}^m$ be a cone. We say that $x \in \mathbb{R}^n$ is an **e-minimizer** *for f if there is a neighbourhood U of x such that*

$$f(y) - f(x) \notin -(C \setminus \{0\}).$$

Example: e-minimizer

$$f : \mathbb{R} \to \mathbb{R}^2; C = \{(x, y); x \geq 0, y \geq 0\}$$



Vector optimization – isolated minimizer of second-order

Definition

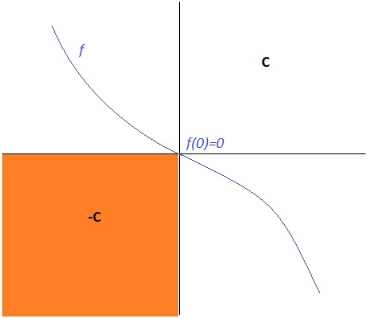
Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function, $C \subset \mathbb{R}^m$ be a cone. We say that $x \in \mathbb{R}^n$ is an **isolated minimizer of second-order** *for f if there is a neighbourhood U of x and a constant $A > 0$ such that*

$$\sup_{\xi \in \Gamma_C} \langle \xi, f(y) - f(x) \rangle \geq A \|y - x\|^2, \quad \forall y \in U.$$

Example: isolated minimizer of second-order

$$C = \{(x,y); x \geq 0, y \geq 0\}, f : \mathbb{R} \to \mathbb{R}^2$$

$$f(t) = \begin{cases} (t, -t^2) & , \quad \text{if } t \geq 0, \\ (t, t^2) & , \quad \text{if } t < 0. \end{cases}$$



Sufficient optimality condition

Theorem 13

Let a function $f : \mathbb{R}^n \to \mathbb{R}^m$ be ℓ–stable at $x \in \mathbb{R}^n$. Let $\triangle(x) := \{\xi \in \mathbb{R}^m : \xi f'(x) = 0, \|\xi\| = 1\} \cap C^* \neq \emptyset$ and suppose that for every $h \in S_{\mathbb{R}^n}$ one of the following two conditions is satisfied:

(i) $f'(x)h \notin -C$,

(ii) $f'(x)h \in -(C \setminus \text{int } C)$ *and*

$\min_{y \in f''_P(x;h)} \max\{\langle \xi, y \rangle : \xi \in \triangle(x) \cap C^*\} > 0.$

Then x is an isolated minimizer of second-order for f.

$$\begin{aligned} f''_P(x;h) &:= \operatorname{Limsup}_{t\downarrow 0}\{(2/t^2)(f(x+th) - f(x) - tf'(x)h)\} \\ &:= \{y \in \mathbb{R}^n : \exists\{t_k\}_{k=1}^{\infty} \text{ such that } t_k \downarrow 0 \text{ and} \\ &\quad (2/t_k^2)(f(x+t_kh) - f(x) - t_kf'(x)h) \to y\}. \end{aligned}$$

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Thank you for your attention.